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Supplement to: Goncharov's Relations in Bloch's higher Chow Group $CH^3(F,5)$

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In this supplement we prove the admissibility of all the cycles appearing in the paper *Goncharov's Relations in Bloch's higher Chow Group* $CH^3(F,5)$. First let's recall the following two Lemmas:

Lemma 3.1. (Gangl-Müller-Stach) *Let* f_i *(*$i = 1,2,3,5$*) be rational functions and* $f_4(x,y)$ *be a product of fractional linear transformations of the form* $(a_1x+b_1y+c_1)/(a_2x+b_2y+c_2)$. *We assume that all the cycles in the lemma are* admissible *and write*

$$Z(f_1,f_2) = [f_1,f_2,f_3,f_4,f_5] = [f_1(x),f_2(y),f_3(x),f_4(x,y),f_5(y)]$$

if no confusion arises.
(i) If $f_4(x,y) = g(x,y)h(x,y)$ *then*

$$[f_1,f_2,f_3,f_4,f_5] = [f_1,f_2,f_3,g,f_5] + [f_1,f_2,f_3,h,f_5].$$

(ii) Assume that $f_1 = f_2$ *and that for each non-constant solution* $y = r(x)$ *of* $f_4(x,y) = 0$ *and* $1/f_4(x,y) = 0$ *one has* $f_2(r(x)) = f_2(x)$.

(a) If $f_3(x) = g(x)h(x)$ *then*

$$[f_1,f_2,f_3,f_4,f_5] = [f_1,f_2,g,f_4,f_5] + [f_1,f_2,h,f_4,f_5].$$

(b) Similarly, if $f_5(y) = g(y)h(y)$ *then*

$$[f_1,f_2,f_3,f_4,f_5] = [f_1,f_2,f_3,f_4,g] + [f_1,f_2,f_3,f_4,h].$$

(c) If $f_1 = f_2 = gh$ *and* $g(r(x)) = g(x)$ *or* $g(r(x)) = h(x)$ *then*

$$2Z(f_1,f_2) = Z(g,f_2) + Z(h,f_2) + Z(f_1,g) + Z(f_1,h) \tag{1}$$

and

$$Z(f_1,f_2) = Z(g,g) + Z(h,h) + Z(h,g) + Z(g,h). \tag{2}$$

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Lemma 3.2. *Assume that f_i, $i = 1,2,3,5$, are rational functions of one variable and p_4 and q_4 are rational functions of two variables. Assume that the only non-constant solution of $p_4(x,y) = 0,\infty$ is $y = x$ and the same for $q_4(x,y)$.*

(i) If $f_3 = gh$ then

$$\begin{aligned}[f_1,f_2,f_3,p_4,f_5] + [f_2,f_1,f_3,q_4,f_5] =&[f_1,f_2,g,p_4,f_5] + [f_2,f_1,g,q_4,f_5]\\ &+[f_1,f_2,h,p_4,f_5] + [f_2,f_1,h,q_4,f_5]\end{aligned}$$

if all cycles are admissible. A similar result holds if $f_5 = gh$.

(ii) If $f_2 = gh$ then

$$\begin{aligned}[f_1,f_2,f_3,p_4,f_5] + [f_2,f_1,f_3,q_4,f_5] =&[f_1,g,f_3,p_4,f_5] + [g,f_1,f_3,q_4,f_5]\\ &+[f_1,h,f_3,p_4,f_5] + [h,f_1,f_3,q_4,f_5]\end{aligned}$$

if all cycles are admissible.

We want to prove the following

Theorem 4.1. *Goncharov's 22 term relations hold in $\mathcal{CH}^3(F,5)$: for any $a,b,c \in \mathbb{P}^1_F$*

$$\begin{aligned}R(a,b,c) = \{-abc\} + \bigoplus_{\mathrm{cyc}(a,b,c)} \Big(&\{ca-a+1\} + \Big\{\frac{ca-a+1}{ca}\Big\} - \Big\{\frac{ca-a+1}{c}\Big\}\\ &+ \Big\{\frac{a(bc-c+1)}{-(ca-a+1)}\Big\} + \Big\{\frac{bc-c+1}{b(ca-a+1)}\Big\} + \{c\} - \Big\{\frac{bc-c+1}{bc(ca-a+1)}\Big\} - \eta\Big) = 0, \quad (3)\end{aligned}$$

where $\mathrm{cyc}(a,b,c)$ *means cyclic permutations of a,b and c, provided that none of terms is $\{0\}$ or $\{1\}$ except for η (non-degeneracy condition).*

Step (1). Construction of $\{k(c)\}$.

Let $f(x) = x$, $A(x) = (ax-a+1)/a$ and $B(x) = bx-x+1$. Let $k(x) = B(x)/abxA(x)$ and $l(y) = 1-(k(c)/k(y))$. By definition

$$\{k(c)\} = \Big[x, y, 1-x, 1-\frac{y}{x}, 1-\frac{k(c)}{y}\Big]$$

which is easy to see as admissible. In the paper we mentioned that for $\mu = -(ab-b+1)/a$

$$4\{k(c)\} = Z\Big(\frac{B}{\mu f A}, \frac{B}{\mu f A}\Big).$$

Here for any two rational functions f_1 and f_2 of one variable we set

$$Z(f_1,f_2) = \Big[f_1(x), f_2(y), 1-k(x), 1-\frac{k(y)}{k(x)}, l(y)\Big].$$

We here need to prove that the following cycles

$$\Big[y, 1-x, 1-\frac{y}{x}, 1-\frac{k(c)}{y}\Big], \qquad \Big[x, 1-x, 1-\frac{y}{x}, 1-\frac{k(c)}{y}\Big], \quad \Big[1-x, 1-\frac{y}{x}, 1-\frac{k(c)}{y}\Big].$$

are admissible and negligible.

$$\boxed{Z_x := [y, 1-x, 1-\frac{y}{x}, 1-\frac{k(c)}{y}]}$$

We have

$$\begin{aligned}\partial_1^0(Z_x) \subset \{t_3 = 1\}, \quad \partial_1^\infty(Z_x) \subset \{t_4 = 1\}, \quad \partial_2^\infty(Z_x) \subset \{t_3 = 1\}, \\ \partial_3^\infty(Z_x) \subset \{t_2 = 1\}, \quad \partial_4^\infty(Z_x) \subset \{t_3 = 1\},\end{aligned}$$

and

$$\partial_2^0(Z_x) = \partial_3^0(Z_x) = \Big[x, 1-x, 1-\frac{k(c)}{x}\Big], \quad \partial_4^0(Z_x) = \Big[k(c), 1-x, 1-\frac{k(c)}{x}\Big],$$

which are both admissible because

$$1 - k(c) = \frac{(c-1)(1+abc)}{abcA(c)} \neq 0. \tag{4}$$

$$\boxed{Z_y := [x, 1-x, 1-\frac{y}{x}, 1-\frac{k(c)}{y}]}$$

We have

$$\begin{aligned}\partial_1^0(Z_y) \subset \{t_2 = 1\}, \quad \partial_1^\infty(Z_y) \subset \{t_3 = 1\}, \quad \partial_2^0(Z_y) \subset \{t_1 = 1\}, \\ \partial_2^\infty(Z_y) \subset \{t_3 = 1\}, \quad \partial_3^\infty(Z_y) \subset \{t_2 = 1\}, \quad \partial_4^\infty(Z_y) \subset \{t_3 = 1\},\end{aligned}$$

and

$$\partial_3^0(Z_y) = \partial_4^0(Z_y) = \Big[x, 1-x, 1-\frac{k(c)}{x}\Big] = \partial_2^0(Z_x)$$

which is admissible.

$$\boxed{Z_{x,y} := [1-x, 1-\frac{y}{x}, 1-\frac{k(c)}{y}]}$$

We have

$$\partial_1^\infty(Z_y) \subset \{t_2 = 1\}, \quad \partial_2^\infty(Z_y) \subset \{t_1 = 1\}, \quad \partial_3^\infty(Z_y) \subset \{t_2 = 1\},$$

and

$$\partial_1^0(Z_x) = \partial_2^0(Z_y) = \partial_3^0(Z_y) = \Big[1-y, 1-\frac{k(c)}{y}\Big]$$

which is admissible by (4).

Step (2). The key reparametrization and a simple expression of $\{k(c)\}$.

Applying Lemma 3.1(ii) we see that

$$\begin{aligned}4\{k(c)\} = Z\Big(\frac{\mu f A}{B}, \frac{\mu f A}{B}\Big) &= Z(A, A) + Z\Big(\frac{\mu f}{B}, A\Big) + Z\Big(A, \frac{\mu f}{B}\Big) + Z\Big(\frac{\mu f}{B}, \frac{\mu f}{B}\Big) \\ &= Z(A, A) + \rho_x Z(A, A) + \rho_y Z(A, A) + \rho_{x,y} Z(A, A) = 4Z(A, A). \end{aligned} \tag{5}$$

We only need to show that the following cycle is admissible:

$$\boxed{Z_A := Z(A,A) = \Big[A(x), A(y), 1-k(x), 1-\frac{k(y)}{k(x)}, l(y)\Big]}$$

Note that

$$1-k(x) = \frac{(x-1)(1+abx)}{abxA(x)}, \tag{6}$$

$$1-\frac{k(y)}{k(x)} = \frac{(y-x)(yB(x)+A(x))}{yA(y)B(x)} = \frac{(y-x)(xB(y)+A(y))}{yA(y)B(x)}. \tag{7}$$

We have

$$\partial_1^0(Z_A) \subset \{t_4=1\}, \quad \partial_1^\infty(Z_A) \subset \{t_3=1\}, \partial_2^0(Z_A) \subset \{t_5=1\}, \quad \partial_2^\infty(Z_A) \subset \{t_4=1\},$$
$$\partial_3^\infty(Z_A) \subset \{t_4=1\}, \quad \partial_4^\infty(Z_A) \subset \{t_3=1\} \cup \{t_5=1\}, \quad \partial_5^\infty(Z_A) \subset \{t_4=1\},$$
$$\partial_3^0(Z_A) = \Big[\frac{1}{a}, A(y), 1-k(y), l(y)\Big] + \Big[A\Big(\frac{-1}{ab}\Big), A(y), 1-k(y), l(y)\Big],$$
$$\partial_4^0(Z_A) = \Big[A(y), A(y), 1-k(y), l(y)\Big] + \Big[\frac{\mu y}{B(y)}, A(y), 1-k(y), l(y)\Big],$$
$$\partial_5^0(Z_A) = \Big[A(x), A(c), 1-k(x), l(x)\Big] + \Big[A(x), A(y_2), 1-k(x), l(x)\Big],$$

where the last equation comes from the two solutions of $l(y)=0$:

$$y_1 = c \quad \text{and} \quad y_2 = -\frac{ac-a+1}{a(bc-c+1)} = -\frac{A(c)}{B(c)} = \rho_c(c). \tag{8}$$

By non-degeneracy assumption and

$$A(y_2) = \rho_c(A(c)) = c\mu/B(c) \neq 0, \infty, \tag{9}$$

$$B(y_2) = \rho_c(B(c)) = -\mu/B(c) \neq 0, \infty, \tag{10}$$

it suffices to show the following cycles are admissible:

$$L := \Big[A(y), 1-k(y), l(y)\Big], \qquad L' := \Big[A(y), A(y), 1-k(y), l(y)\Big],$$
$$L'' := \Big[\frac{\mu y}{B(y)}, A(y), 1-k(y), l(y)\Big].$$

- L is admissible. Because $l(y) = 1 - yB(c)A(y)/cA(c)B(y)$ we have

  $$\partial_1^0(L) \subset \{t_3=1\}, \quad \partial_1^\infty(L) \subset \{t_2=1\}, \quad \partial_2^\infty(L) \subset \{t_3=1\}, \quad \partial_3^\infty(L) \subset \{t_2=1\}.$$

  Moreover, by non-degeneracy assumption we see that (note that $k(1) = k(-1/ab) = 1$ by (6))

  $$A(1) = \frac{1}{a} \neq 0, \infty, \quad l(1) = 1-k(c) = \frac{(c-1)(1+abc)}{bc(ca-a+1)} \neq 0, \infty, \tag{11}$$

  $$A\Big(\frac{-1}{ab}\Big) = \frac{\mu}{b} \neq 0, \infty, \quad l\Big(\frac{-1}{ab}\Big) = 1-k(c) \neq 0, \infty, \tag{12}$$

  $$aby_2 + 1 = \frac{(1-c)(ab-b+1)}{bc-c+1} \neq 0, \infty. \tag{13}$$

Thus both $\partial_2^0(L) = [A(1), l(1)] + [A(-1/ab), l(-1/ab)]$ and $\partial_3^0(L) = [A(c), 1-k(c)] + [A(y_2), 1-k(c)]$ are clearly admissible by non-degeneracy assumption, (11), (12), and (13).

- L' is admissible. This follows from the above proof for L.
- L'' is admissible. This also follows from the proof for L because $\mu y/B(y) \neq 0, \infty$ when $y = 1, -1/ab, c, y_2$ by (10).

Step (3). Some admissible cycles for decomposition of $\{k(c)\}$.

Define the following cycles

$$\begin{aligned}
Z_1(A,A) =& \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y)}{\mu}, \frac{x-1}{x}, \frac{y-x}{A(y)}, l(y)\Big] \\
Z_1 =& \Big[A(x), A(y), \frac{x-1}{x}, \frac{y-x}{A(y)}, l(y)\Big], \\
Z_2(A,A) =& \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y)}{\mu}, \frac{x-1}{x}, \Big(\frac{A(y)}{y}\Big)\Big(1-\frac{\mu x}{A(y)B(x)}\Big), l(y)\Big] \\
Z_2 =& \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y)}{\mu}, \frac{x-1}{x}, 1-\frac{\mu x}{A(y)B(x)}, l(y)\Big], \\
Z_3(A,A) =& \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y)}{\mu}, \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, l(y)\Big] \\
Z_3 =& \Big[A(x), A(y), \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, l(y)\Big], \\
Z_4(A,A) =& \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y)}{\mu}, \frac{abx+1}{abA(x)}, \Big(\frac{A(y)}{y}\Big)\Big(1-\frac{\mu x}{A(y)B(x)}\Big), l(y)\Big] \\
Z_4 =& \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y)}{\mu}, \frac{abx+1}{aA(x)}, \Big(\frac{A(y)}{y}\Big)\Big(1-\frac{\mu x}{A(y)B(x)}\Big), l(y)\Big].
\end{aligned}$$

Claim 1. Modulo admissible and negligible cycles the two cycles $Z_1(A,A)$ and Z_1 are the same and both admissible.

$$\boxed{Z_1 = \Big[A(x), A(y), \frac{x-1}{x}, \frac{y-x}{A(y)}, l(y)\Big]}$$

On $\partial_1^0(Z_1)$ we have $x = 1 - a^{-1}$ and $(y-x)/A(y) = 1$. By similar argument we get

$$\begin{aligned}
\partial_1^0(Z_1) \subset& \{t_4 = 1\}, \quad \partial_1^\infty(Z_1) \subset \{t_3 = 1\}, \quad \partial_2^0(Z_1) \subset \{t_5 = 1\}, \\
\partial_2^\infty(Z_1) \subset& \{t_4 = 1\}, \quad \partial_4^\infty(Z_1) \subset \{t_5 = 1\}.
\end{aligned}$$

So we still need to show the following cycles are admissible:

$$\begin{aligned}
\partial_3^0(Z_1) =& \Big[\frac{1}{a}, A(y), \frac{y-1}{A(y)}, l(y)\Big], \qquad \partial_3^\infty(Z_1) = \Big[\frac{1-a}{a}, A(y), \frac{y}{A(y)}, l(y)\Big], \\
\partial_4^0(Z_1) =& \Big[A(y), A(y), \frac{y-1}{y}, l(y)\Big], \quad \partial_5^\infty(Z_1) = \Big[A(x), \frac{\mu}{b-1}, \frac{x-1}{x}, \frac{B(x)}{-\mu}\Big], \\
\partial_5^0(Z_1) =& \Big[A(x), A(c), \frac{x-1}{x}, \frac{c-x}{A(c)}\Big] + \Big[A(x), A(y_2), \frac{x-1}{x}, \frac{y_2-x}{A(y_2)}\Big].
\end{aligned}$$

So we still need to show the following cycles are admissible:

$$\begin{aligned} T :=& [A(y), (y-1)/A(y), l(y)], & U :=& [A(y), y/A(y), l(y)], \\ V :=& [A(y), A(y), 1-y^{-1}, l(y)], & W :=& \Big[A(x), \frac{x-1}{x}, \frac{B(x)}{-\mu}\Big], \\ X :=& \Big[A(x), \frac{x-1}{x}, \frac{c-x}{A(c)}\Big], & Y :=& \Big[A(x), \frac{x-1}{x}, \frac{y_2-x}{A(y_2)}\Big]. \end{aligned}$$

- T is admissible. Because $l(y) = 1 - \frac{yB(c)A(y)}{cA(c)B(y)}$ we have

  $$\partial_1^0(T) \subset \{t_3 = 1\}, \quad \partial_1^\infty(T) \subset \{t_2 = 1\}, \quad \partial_2^\infty(T) \subset \{t_3 = 1\}.$$

  Moreover $\partial_2^0(T) = [1/a, l(1)]$ and $\partial_3^\infty(T) = [\mu/(b-1), -b/\mu]$ are clearly admissible by non-degeneracy assumption and (11). Lastly, from the two solutions of $l(y) = 0$ in (8) and (9) we get

  $$\partial_3^0(T) = \Big[A(c), \frac{c-1}{A(c)}\Big] + \Big[A(y_2), \frac{y_2-1}{A(y_2)}\Big]$$

  which is admissible by non-degeneracy assumption.

- U is admissible. Similar to T we have

  $$\partial_1^0(U) \subset \{t_3 = 1\}, \quad \partial_1^\infty(U) \subset \{t_2 = 1\}, \quad \partial_2^0(U) \subset \{t_3 = 1\}, \quad \partial_2^\infty(U) \subset \{t_3 = 1\}.$$

  Moreover, $\partial_3^\infty(U) = [\mu/(b-1), a/(ab-b+1)]$ is clearly admissible by non-degeneracy assumption. Lastly, from (8) and (9) we get

  $$\partial_3^0(U) = \Big[A(c), \frac{c}{A(c)}\Big] + \Big[A(y_2), \frac{y_2}{A(y_2)}\Big]$$

  which is admissible by non-degeneracy assumption.

- V is admissible. First it's easy to see that

  $$\begin{aligned} &\partial_1^0(V) \subset \{t_4 = 1\}, \quad \partial_1^\infty(V) \subset \{t_3 = 1\}, \quad \partial_2^0(V) \subset \{t_4 = 1\}, \quad \partial_2^\infty(V) \subset \{t_3 = 1\}, \\ &\partial_3^0(V) = [1/a, 1/a, l(1)], \quad \partial_3^\infty(V) \subset \{t_4 = 1\}, \quad \partial_4^\infty(V) = \Big[\frac{\mu}{b-1}, \frac{\mu}{b-1}, b\Big]. \end{aligned}$$

  From (11) both cycles are clearly admissible by non-degeneracy assumption. Lastly, from the two solutions of $l(y) = 0$ in (8) we get

  $$\partial_4^0(V) = \Big[A(c), A(c), \frac{c-1}{c}\Big] + \Big[A(y_2), A(y_2), \frac{y_2-1}{y_2}\Big]$$

  which is admissible by (9) and

  $$\frac{y_2-1}{y_2} = \frac{1+abc}{ac-a+1} \neq \infty, 0. \tag{14}$$

- W is admissible. We can compute as follows:

$$\partial_1^0(W) \subset \{t_3 = 1\}, \qquad \partial_1^\infty(W) \subset \{t_2 = 1\}, \quad \partial_2^0(W) = \Big[\frac{1}{a}, \frac{-b}{\mu}\Big],$$
$$\partial_2^\infty(W) = \Big[\frac{1-a}{a}, \frac{-1}{\mu}\Big], \quad \partial_3^0(W) = \Big[-\mu, b\Big], \quad \partial_3^\infty(W) \subset \{t_2 = 1\}.$$

All the cycles above are clearly admissible.

- X is admissible. Similar to W we have

$$\partial_1^0(X) \subset \{t_3 = 1\}, \qquad \partial_1^\infty(X) \subset \{t_2 = 1\}, \qquad \partial_2^0(X) = \Big[\frac{1}{a}, \frac{c-1}{A(c)}\Big],$$
$$\partial_2^\infty(X) = \Big[\frac{1-a}{a}, \frac{c}{A(c)}\Big], \quad \partial_3^0(X) = \Big[A(c), \frac{c-1}{c}\Big], \quad \partial_3^\infty(X) \subset \{t_2 = 1\}.$$

All the cycles above are clearly admissible.

- Y is admissible. Similar to the above we get

$$\partial_1^0(Y) \subset \{t_3 = 1\}, \qquad \partial_1^\infty(Y) \subset \{t_2 = 1\}, \qquad \partial_2^0(Y) = \Big[\frac{1}{a}, \frac{y_2-1}{A(y_2)}\Big],$$
$$\partial_2^\infty(Y) = \Big[\frac{1-a}{a}, \frac{y_2}{A(y_2)}\Big], \quad \partial_3^0(Y) = \Big[A(y_2), \frac{y_2-1}{y_2}\Big], \quad \partial_3^\infty(Y) \subset \{t_2 = 1\}.$$

By (8), (14) and (9) all the cycles above are clearly admissible. This concludes the proof that Z_1 is an admissible cycle.

$$\boxed{Z_1(A,A) = \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y)}{\mu}, \frac{x-1}{x}, \frac{y-x}{A(y)}, l(y)\Big]}$$

Throughout the proof that Z_1 is an admissible cycle we never use the hyperplane $\{t_1 = 1\}$ and moreover $(b-1)/\mu \neq 0, \infty$ by non-degeneracy assumption. Therefore we can use exactly the same proof to show the admissibility of $Z_1(A,A)$.

$$\boxed{Z_{11} = \Big[\frac{b-1}{\mu}, A(y), \frac{x-1}{x}, \frac{y-x}{A(y)}, l(y)\Big] \in C^1(F,1) \wedge C^2(F,4)}$$

Let

$$Z'_{11} = \Big[A(y), \frac{x-1}{x}, \frac{y-x}{A(y)}, l(y)\Big]$$

It is easy to see that

$$\partial_1^0(Z'_{11}) \subset \{t_4 = 1\}, \qquad \partial_1^\infty(Z'_{11}) \subset \{t_3 = 1\},$$
$$\partial_2^0(Z'_{11}) = \Big[A(y), \frac{y-1}{A(y)}, l(y)\Big] = T, \quad \partial_2^\infty(Z'_{11}) = \Big[A(y), \frac{y}{A(y)}, l(y)\Big] = U,$$
$$\partial_3^0(Z'_{11}) = \Big[A(y), \frac{y-1}{y}, l(y)\Big] =: V', \quad \partial_3^\infty(Z'_{11}) \subset \{t_4 = 1\},$$
$$\partial_4^0(Z'_{11}) = \Big[A(c), \frac{x-1}{x}, \frac{c-x}{A(c)}\Big] + \Big[A(y_2), \frac{x-1}{x}, \frac{y_2-x}{A(y_2)}\Big] =: X' + Y',$$
$$\partial_4^\infty(Z'_{11}) = \Big[\frac{\mu}{b-1}, \frac{x-1}{x}, \frac{B(x)}{-\mu}\Big] =: W'.$$

The admissibility of V', X', Y' and W' follows from the proof of that of V, X, Y and W, respectively.

$$\boxed{Z_{12} = \left[A(x), \frac{b-1}{\mu}, \frac{x-1}{x}, \frac{y-x}{A(y)}, l(y)\right] \in C^1(F,1) \wedge C^2(F,4)}$$

Let

$$Z_{12}' = \left[A(x), \frac{x-1}{x}, \frac{y-x}{A(y)}, l(y)\right]$$

It is easy to see that

$$\begin{aligned}
&\partial_1^0(Z_{12}') \subset \{t_3 = 1\}, \quad \partial_1^\infty(Z_{12}') \subset \{t_2 = 1\},\\
&\partial_2^0(Z_{12}') = \left[\frac{1}{a}, \frac{y-1}{A(y)}, l(y)\right] = T', \quad \partial_2^\infty(Z_{12}') = \left[\frac{1-a}{a}, \frac{y}{A(y)}, l(y)\right] = U',\\
&\partial_3^0(Z_{12}') = \left[A(y), \frac{y-1}{y}, l(y)\right] = V', \quad \partial_3^\infty(Z_{12}') \subset \{t_4 = 1\},\\
&\partial_4^0(Z_{12}') = \left[A(x), \frac{x-1}{x}, \frac{c-x}{A(c)}\right] + \left[A(x), \frac{x-1}{x}, \frac{y_2-x}{A(y_2)}\right] = X + Y,\\
&\partial_4^\infty(Z_{12}') = \left[A(x), \frac{x-1}{x}, \frac{B(x)}{-\mu}\right] = W.
\end{aligned}$$

The admissibility of T', U' and V' follows from the proof of that of T, U and V, respectively.

$$\boxed{Z_{13} = \left[\frac{b-1}{\mu}, \frac{b-1}{\mu}, \frac{x-1}{x}, \frac{y-x}{A(y)}, l(y)\right] \in C^1(F,1) \wedge C^1(F,1) \wedge C^1(F,3)}$$

Let

$$Z_{13}' = \left[\frac{x-1}{x}, \frac{y-x}{A(y)}, l(y)\right]$$

It is easy to see that $\partial_2^\infty(Z_{13}') \subset \{t_3 = 1\}$ and

$$\begin{aligned}
&\partial_1^0(Z_{13}') = \left[\frac{y-1}{A(y)}, l(y)\right] =: T'', \quad \partial_1^\infty(Z_{13}') = \left[\frac{y}{A(y)}, l(y)\right] =: U'',\\
&\partial_2^0(Z_{13}') = \left[\frac{y-1}{y}, l(y)\right] =: V'', \quad \partial_3^\infty(Z_{13}') = \left[\frac{x-1}{x}, \frac{B(x)}{-\mu}\right] =: W'',\\
&\partial_3^0(Z_{13}') = \left[\frac{x-1}{x}, \frac{c-x}{A(c)}\right] + \left[\frac{x-1}{x}, \frac{y_2-x}{A(y_2)}\right] =: X'' + Y''.
\end{aligned}$$

The admissibility of T'', U'', V'', X'', Y'' and W'' is easy to check. It also follows from the proof of the admissibility of T, U, V, X, Y and W, respectively.

All the above justifies the use of Lemma 3.1(ii)(c)(2) to get:

$$Z_1(A, A) = Z_1 + Z_{11} + Z_{12} + Z_{13}.$$

Claim 2. Modulo admissible and negligible cycles the two cycles $Z_2(A, A)$ and Z_2 are the same and both admissible.

$$\boxed{Z_2 = \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y)}{\mu}, \frac{x-1}{x}, 1-\frac{\mu x}{A(y)B(x)}, l(y)\Big]}$$

It's not hard to see that

$$\begin{aligned}
&\partial_1^\infty(Z_2) \subset \{t_3 = 1\}, \quad \partial_2^0(Z_2) \subset \{t_5 = 1\}, \quad && \partial_2^\infty(Z_2) \subset \{t_4 = 1\},\\
&\partial_3^\infty(Z_2) \subset \{t_4 = 1\}, \quad \partial_4^\infty(Z_2) \subset \{t_1 = 1\} \cup \{t_5 = 1\}, \quad && \partial_5^\infty(Z_2) \subset \{t_2 = 1\},
\end{aligned}$$

and

$$\begin{aligned}
\partial_1^0(Z_2) &= \Big[\frac{(b-1)A(y)}{\mu}, \frac{1}{1-a}, \frac{y}{A(y)}, l(y)\Big] =: U''',\\
\partial_3^0(Z_2) &= \Big[\frac{b-1}{a\mu}, \frac{(b-1)A(y)}{\mu}, \frac{aby+1}{abA(y)}, l(y)\Big],\\
\partial_4^0(Z_2) &= \Big[\frac{(b-1)y}{B(y)}, \frac{(b-1)A(y)}{\mu}, \frac{aby+1}{abA(y)}, l(y)\Big],\\
\partial_5^0(Z_2) &= \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(c)}{\mu}, \frac{x-1}{x}, \frac{y_2-x}{y_2B(x)}\Big]\\
&\quad + \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y_2)}{\mu}, \frac{x-1}{x}, \frac{c-x}{cB(x)}\Big].
\end{aligned}$$

Then U''' is admissible similar to U. By non-degeneracy assumption it suffices to show the following cycles are admissible:

$$\begin{aligned}
&P := \Big[\frac{(b-1)A(y)}{\mu}, \frac{aby+1}{abA(y)}, l(y)\Big], \quad Q := \Big[\frac{(b-1)y}{B(y)}, \frac{(b-1)A(y)}{\mu}, \frac{aby+1}{abA(y)}, l(y)\Big],\\
&R := \Big[\frac{(b-1)A(x)}{\mu}, \frac{x-1}{x}, \frac{y_2-x}{y_2B(x)}\Big], \quad S := \Big[\frac{(b-1)A(x)}{\mu}, \frac{x-1}{x}, \frac{c-x}{cB(x)}\Big].
\end{aligned}$$

- P is admissible. We have

$$\begin{aligned}
&\partial_1^0(P) \subset \{t_3 = 1\}, \quad \partial_1^\infty(P) \subset \{t_2 = 1\}, \quad \partial_2^\infty(P) \subset \{t_3 = 1\}, \quad \partial_3^\infty(P) \subset \{t_1 = 1\},\\
&\partial_2^0(P) = \Big[\frac{b-1}{b}, l\Big(\frac{-1}{ab}\Big)\Big], \quad \partial_3^0(P) = \Big[\frac{(b-1)A(c)}{\mu}, \frac{abc+1}{abA(c)}\Big] + \Big[\frac{(b-1)A(y_2)}{\mu}, \frac{aby_2+1}{abA(y_2)}\Big].
\end{aligned}$$

  All the cycles above are admissible by (12), (14), (9) and (13).

- Q is admissible. First it's easy to see that

$$\begin{aligned}
&\partial_1^0(Q) \subset \{t_4 = 1\}, \quad \partial_1^\infty(Q) \subset \{t_2 = 1\}, \quad \partial_2^0(Q) \subset \{t_4 = 1\}, \quad \partial_2^\infty(Q) \subset \{t_3 = 1\},\\
&\partial_3^0(Q) = [(1-b)/(ab-b+1), (b-1)/b, l(-1/ab)], \quad \partial_3^\infty(Q) \subset \{t_4 = 1\},\\
&\partial_4^0(Q) = \Big[\frac{(b-1)c}{B(c)}, \frac{(b-1)A(c)}{\mu}, \frac{abc+1}{abA(c)}\Big] + \Big[\frac{(b-1)y_2}{B(y_2)}, \frac{(b-1)A(y_2)}{\mu}, \frac{aby_2+1}{abA(y_2)}\Big],\\
&\partial_4^\infty(Q) = [\mu/(b-1), \mu/(b-1), b].
\end{aligned}$$

  All the cycles in the above are admissible by (9), (14), (12) and (13) and (10).

- R is admissible. This is because that the zeros and poles of the three coordinate functions are all distinct.
- S is admissible. Same as R.

This concludes the proof that Z_2 is an admissible cycle.

$$\boxed{Z_2(A,A)=\Big[\frac{(b-1)A(x)}{\mu},\frac{(b-1)A(y)}{\mu},\frac{x-1}{x},\Big(\frac{A(y)}{y}\Big)\Big(1-\frac{\mu x}{A(y)B(x)}\Big),l(y)\Big]}$$

We can use exactly the same proof for Z_2 except the following modifications.

First, we need to look for places where we used $\{t_4=1\}$. Then only the following needs to be re-considered:

$$\partial_3^\infty(Z_2(A,A))=\Big[\frac{(1-b)(1-a)}{ab-b+1},\frac{(b-1)A(y)}{\mu},\frac{A(y)}{y},l(y)\Big]$$

which can be checked to be admissible as follows: Let

$$N=\Big[\frac{(b-1)A(y)}{\mu},\frac{A(y)}{y},l(y)\Big]. \tag{15}$$

Then

$$\partial_1^0(N)\subset\{t_3=1\},\quad \partial_1^\infty(N)\subset\{t_2=1\},\quad \partial_2^0(N)\subset\{t_3=1\},\quad \partial_2^\infty(N)\subset\{t_3=1\},$$
$$\partial_3^0(N)=\Big[\frac{(b-1)A(c)}{\mu},\frac{A(c)}{c}\Big]+\Big[\frac{(b-1)A(y_2)}{\mu},\frac{A(y_2)}{y_2}\Big],\quad \partial_3^\infty(N)\subset\{t_1=1\}.$$

This shows that N, hence $\partial_3^\infty(Z_2(A,A))$, is admissible.

Second, R (resp. S) is still admissible if we multiply the third coordinate by $A(c)/c$ (resp. $A(y_2)/y_2$) because the three coordinate functions of the modified cycle still have distinct zeros and poles.

$$\boxed{Z_{21}=\Big[\frac{(b-1)A(x)}{\mu},\frac{(b-1)A(y)}{\mu},\frac{x-1}{x},\Big(\frac{A(y)}{y}\Big),l(y)\Big]\in C^1(F,2)\wedge C^2(F,3)}$$

This cycle is product of two admissible cycles $[(b-1)A(x)/\mu,(x-1)/x]$ and N given by (15).

All the above justifies the use of Lemma 3.1(i) to get

$$Z_2(A,A)=Z_2+Z_{21}.$$

Claim 3. Modulo admissible and negligible cycles the two cycles $Z_3(A,A)$ and Z_3 are the same and both admissible.

$$\boxed{Z_3=\Big[A(x),A(y),\frac{abx+1}{abA(x)},\frac{y-x}{A(y)},l(y)\Big]}$$

It's not hard to see that

$$\begin{aligned}\partial_1^0(Z_3) \subset& \{t_4=1\}, \quad \partial_1^\infty(Z_3) \subset \{t_3=1\}, \quad \partial_2^0(Z_3) \subset \{t_5=1\},\\ \partial_2^\infty(Z_3) \subset& \{t_4=1\}, \quad \partial_3^\infty(Z_3) \subset \{t_4=1\}, \quad \partial_4^\infty(Z_3) \subset \{t_5=1\}.\end{aligned}$$

So we still need to show the following cycles are admissible:

$$\begin{aligned}\partial_3^0(Z_1) =& \Big[A(-1/ab), A(y), \frac{aby+1}{abA(y)}, l(y)\Big], \quad \partial_4^0(Z_1) = \Big[A(y), A(y), \frac{aby+1}{abA(y)}, l(y)\Big],\\ \partial_5^\infty(Z_1) =& \Big[A(x), \frac{\mu}{b-1}, \frac{abx+1}{abA(x)}, \frac{B(x)}{-\mu}\Big],\\ \partial_5^0(Z_1) =& \Big[A(x), A(c), \frac{abx+1}{abA(x)}, \frac{a(c-x)}{ac-a+1}\Big] + \Big[A(x), A(y_2), \frac{abx+1}{abA(x)}, \frac{y_2-x}{A(y_2)}\Big]\end{aligned}$$

By non-degeneracy assumption it suffices to show the following cycles are admissible:

$$\begin{aligned}C' :=& \Big[A(y), \frac{aby+1}{abA(y)}, l(y)\Big], \quad C := \Big[A(y), A(y), \frac{aby+1}{abA(y)}, l(y)\Big], \quad D := \Big[A(x), \frac{abx+1}{abA(x)}, \frac{B(x)}{-\mu}\Big],\\ E :=& \Big[A(x), \frac{abx+1}{abA(x)}, \frac{c-x}{A(c)}\Big], \quad F := \Big[A(x), \frac{abx+1}{abA(x)}, \frac{y_2-x}{A(y_2)}\Big].\end{aligned}$$

These are all admissible by easy computations. The only non-obvious one identity is $y_2 - (a-1)/a = A(y_2)$ which is used to show that F is admissible.

$$\boxed{Z_3(A,A) = \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y)}{\mu}, \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, l(y)\Big]}$$

Throughout the proof that Z_3 is an admissible cycle we never used the hyperplane $\{t_1=1\}$ or $\{t_2=1\}$ and moreover $(b-1)/\mu \neq 0, \infty$ by non-degeneracy assumption. Therefore we can use exactly the same proof to show the admissibility of $Z_3(A,A)$.

$$\boxed{Z_{31} = \Big[\frac{b-1}{\mu}, A(y), \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, l(y)\Big] \in C^1(F,1) \wedge C^2(F,4)}$$

Let

$$Z_{31}' = \Big[A(y), \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, l(y)\Big]$$

Then

$$\partial_1^0(Z_{31}') \subset \{t_4=1\}, \quad \partial_1^\infty(Z_{31}') \subset \{t_3=1\}, \quad \partial_2^\infty(Z_{31}') \subset \{t_3=1\}, \quad \partial_3^\infty(Z_{31}') \subset \{t_4=1\},$$

and

$$\begin{aligned}\partial_2^0(Z_{31}') &= \Big[A(y), \frac{aby+1}{abA(y)}, l(y)\Big] = C', \quad \partial_3^0(Z_{31}') = \Big[A(y), \frac{aby+1}{abA(y)}, l(y)\Big] = C',\\ \partial_4^0(Z_{31}') &= \Big[A(c), \frac{abx+1}{abA(x)}, \frac{c-x}{A(c)}\Big] + \Big[A(y_2), \frac{abx+1}{abA(x)}, \frac{y_2-x}{A(y_2)}\Big] =: E' + F',\\ \partial_4^\infty(Z_{31}') &= \Big[\frac{\mu}{b-1}, \frac{abx+1}{abA(x)}, \frac{B(x)}{-\mu}\Big] =: D'.\end{aligned}$$

The cycles E', F' and D' are admissible because the coordinate functions have different zeros and poles.

$$\boxed{Z_{32} = \Big[A(x), \frac{b-1}{\mu}, \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, l(y)\Big] \in C^1(F,1) \wedge C^2(F,4)}$$

Let

$$Z'_{32} = \Big[A(x), \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, l(y)\Big]$$

Then

$$\partial_1^0(Z'_{32}) \subset \{t_3 = 1\}, \quad \partial_1^\infty(Z'_{32}) \subset \{t_2 = 1\}, \quad \partial_2^\infty(Z'_{32}) \subset \{t_3 = 1\}, \quad \partial_3^\infty(Z'_{32}) \subset \{t_4 = 1\},$$

and

$$\partial_2^0(Z'_{32}) = \Big[A\Big(\frac{-1}{ab}\Big), \frac{aby+1}{abA(y)}, l(y)\Big] =: C'', \quad \partial_3^0(Z'_{32}) = \Big[A(y), \frac{aby+1}{abA(y)}, l(y)\Big] = C',$$
$$\partial_4^0(Z'_{32}) = \Big[A(x), \frac{abx+1}{abA(x)}, \frac{c-x}{A(c)}\Big] + \Big[A(x), \frac{abx+1}{abA(x)}, \frac{y_2-x}{A(y_2)}\Big] = E + F,$$
$$\partial_4^\infty(Z'_{32}) = \Big[A(x), \frac{abx+1}{abA(x)}, \frac{B(x)}{-\mu}\Big] = D.$$

The coordinate functions of C'' are all distinct because of (14), (9) and (13) so that all the above cycles are admissible.

$$\boxed{Z_{33} = \Big[\frac{b-1}{\mu}, \frac{b-1}{\mu}, \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, l(y)\Big] \in C^1(F,1) \wedge C^1(F,1) \wedge C^1(F,3)}$$

Let

$$Z'_{33} = \Big[\frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, l(y)\Big]$$

Then

$$\partial_1^0(Z'_{33}) = \Big[\frac{aby+1}{abA(y)}, l(y)\Big] =: C''', \qquad \partial_1^\infty(Z'_{33}) \subset \{t_2 = 1\},$$
$$\partial_2^0(Z'_{33}) = \Big[\frac{aby+1}{abA(y)}, l(y)\Big] = C''', \quad \partial_2^\infty(Z'_{33}) \subset \{t_3 = 1\},$$
$$\partial_3^0(Z'_{33}) = \Big[\frac{abx+1}{abA(x)}, \frac{c-x}{A(c)}\Big] + \Big[\frac{abx+1}{abA(x)}, \frac{y_2-x}{A(y_2)}\Big] =: E'' + F'',$$
$$\partial_3^\infty(Z'_{33}) = \Big[\frac{abx+1}{abA(x)}, \frac{B(x)}{-\mu}\Big] =: D''.$$

The admissibility of C''', E'', F'' and D'' is easy to check.

All the above justifies the use of Lemma 3.1(ii)(c)(2) to get:

$$Z_3(A,A) = Z_3 + Z_{31} + Z_{32} + Z_{33}.$$

Claim 4. Modulo admissible and negligible cycles the two cycles $Z_4(A, A)$ and Z_4 are the same and both admissible.

$$\boxed{Z_4 = \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y)}{\mu}, \frac{abx+1}{aA(x)}, \Big(\frac{A(y)}{y}\Big)\Big(1 - \frac{\mu x}{A(y)B(x)}\Big), l(y)\Big]}$$

Note that

$$\Big(\frac{A(y)}{y}\Big)\Big(1 - \frac{\mu x}{A(y)B(x)}\Big) = \frac{xB(y) + A(y)}{yB(x)} = \frac{yB(x) + A(x)}{yB(x)}. \tag{16}$$

It's not hard to see that

$$\begin{aligned} &\partial_1^0(Z_4) \subset \{t_4 = 1\}, \quad \partial_2^0(Z_4) \subset \{t_5 = 1\}, \quad \partial_2^\infty(Z_4) \subset \{t_4 = 1\}, \\ &\partial_3^\infty(Z_4) \subset \{t_4 = 1\}, \quad \partial_4^\infty(Z_4) \subset \{t_1 = 1\} \cup \{t_5 = 1\}, \quad \partial_5^\infty(Z_4) \subset \{t_1 = 1\}, \end{aligned}$$

and

$$\begin{aligned} \partial_1^\infty(Z_4) &= \Big[\frac{(b-1)A(y)}{\mu}, b, \frac{B(y)}{(b-1)y}, l(y)\Big], \quad \partial_3^0(Z_4) = \Big[\frac{b-1}{b}, \frac{(b-1)A(y)}{\mu}, \frac{y-1}{y}, l(y)\Big], \\ \partial_4^0(Z_4) &= \Big[\frac{(b-1)y}{B(y)}, \frac{(b-1)A(y)}{\mu}, \frac{y-1}{y}, l(y)\Big], \\ \partial_5^0(Z_4) &= \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(c)}{\mu}, \frac{abx+1}{aA(x)}, \frac{xB(c)+A(c)}{cB(x)}\Big] \\ &\quad + \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y_2)}{\mu}, \frac{abx+1}{aA(x)}, \frac{xB(y_2)+A(y_2)}{y_2B(x)}\Big]. \end{aligned}$$

By non-degeneracy assumption it suffices to show the following cycles are admissible:

$$\begin{aligned} G &:= \Big[\frac{(b-1)A(y)}{\mu}, \frac{B(y)}{(b-1)y}, l(y)\Big], \quad H := \Big[\frac{(b-1)A(y)}{\mu}, \frac{y-1}{y}, l(y)\Big], \\ I &:= \Big[\frac{(b-1)y}{B(y)}, \frac{(b-1)A(y)}{\mu}, \frac{y-1}{y}, l(y)\Big], \quad J := \Big[\frac{(b-1)A(x)}{\mu}, \frac{abx+1}{aA(x)}, \frac{xB(c)+A(c)}{cB(x)}\Big] \\ K &:= \Big[\frac{(b-1)A(x)}{\mu}, \frac{abx+1}{aA(x)}, \frac{xB(y_2)+A(y_2)}{y_2B(x)}\Big]. \end{aligned}$$

- G is admissible. We have

  $$\begin{aligned} &\partial_1^0(G) \subset \{t_3 = 1\}, \quad \partial_1^\infty(G) \subset \{t_2 = 1\}, \quad \partial_2^0(G) \subset \{t_1 = 1\}, \quad \partial_2^\infty(G) \subset \{t_3 = 1\}, \\ &\partial_3^0(G) = \Big[\frac{(b-1)A(c)}{\mu}, \frac{B(c)}{(b-1)c}\Big] + \Big[\frac{(b-1)A(y_2)}{\mu}, \frac{B(y_2)}{(b-1)y_2}\Big], \quad \partial_3^\infty(G) \subset \{t_1 = 1\}. \end{aligned}$$

  It follows from (14), (9) and (13) that $\partial_3^0(G)$ is admissible.

- H is admissible. The three coordinate functions have distinct zeros and poles because of (14) and (9).

- I is admissible. We have

$$\begin{aligned}
&\partial_1^0(I) \subset \{t_4 = 1\}, \quad \partial_1^\infty(I) \subset \{t_2 = 1\}, \quad \partial_2^0(I) \subset \{t_4 = 1\}, \quad \partial_2^\infty(I) \subset \{t_1 = 1\}, \\
&\partial_3^0(I) = \Big[\frac{b-1}{b}, \frac{b-1}{a\mu}, l(1)\Big], \quad \partial_3^\infty(I) \subset \{t_4 = 1\}, \quad \partial_4^\infty(I) \subset \{t_2 = 1\} \\
&\partial_4^0(I) = \Big[\frac{(b-1)c}{B(c)}, \frac{(b-1)A(c)}{\mu}, \frac{c-1}{c}\Big] + \Big[\frac{(b-1)y_2}{B(y_2)}, \frac{(b-1)A(y_2)}{\mu}, \frac{y_2-1}{y_2}\Big].
\end{aligned}$$

  All the cycles are admissible by (9), (10), (11) and (14).

- J and K are admissible. By considering the zeros and poles of the coordinate functions it is easy to see that the only nontrivial thing is to check that $\partial_1^0(J) \subset \{t_3 = 1\}$ and $\partial_2^\infty(K) \subset \{t_3 = 1\}$ which follows from equation (16). For example, the zero of the 3rd coordinate of J (resp. K) is $-A(c)/B(c) = y_2$ (resp. $-A(y_2)/B(y_2) = c$).

This concludes the proof that Z_4 is an admissible cycle.

$$\boxed{Z_4(A,A) = \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y)}{\mu}, \frac{abx+1}{abA(x)}, \Big(\frac{A(y)}{y}\Big)\Big(1 - \frac{\mu x}{A(y)B(x)}\Big), l(y)\Big]}$$

We can use exactly the same proof for Z_4 except that instead of I, J and K we need to show I', J' and K' are admissible where

$$\begin{aligned}
I' &:= \Big[\frac{(b-1)y}{B(y)}, \frac{(b-1)A(y)}{\mu}, \frac{y-1}{by}, l(y)\Big], \\
J' &:= \Big[\frac{(b-1)A(x)}{\mu}, \frac{abx+1}{abA(x)}, \frac{xB(c)+A(c)}{cB(x)}\Big] \\
K' &:= \Big[\frac{(b-1)A(x)}{\mu}, \frac{abx+1}{abA(x)}, \frac{xB(y_2)+A(y_2)}{y_2B(x)}\Big].
\end{aligned}$$

Exactly the same proofs are valid because in the proof of I we didn't use the hyperplane $\{t_3 = 1\}$ while for J and K we didn't use $\{t_2 = 1\}$.

$$\boxed{Z_{41} = \Big[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y)}{\mu}, \frac{1}{b}, \Big(\frac{A(y)}{y}\Big)\Big(1 - \frac{\mu x}{A(y)B(x)}\Big), l(y)\Big] \in C^1(F,1) \wedge C^2(F,4)}$$

The same argument for $Z_4(A,A)$ goes through without any problem.

All the above justifies the use of Lemma 3.1(ii)(a) to get

$$Z_4(A,A) = Z_4 + Z_{41}.$$

Step (4). Decomposition of $\rho_x Z_2(A,A) + \rho_y Z_4(A,A) = X_1 - X_2$.

We need to show

$$X_1 = X_{11} + X_{12}, \quad X_2 = X_{21} + X_{22},$$

where

$$
\begin{aligned}
X_{11} =& \Big[\frac{(b-1)x}{B(x)}, \frac{A(y)}{-\mu y}, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l(y)\Big] \\
X_{12} =& \Big[\frac{A(x)}{-\mu x}, \frac{(b-1)y}{B(y)}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l(y)\Big], \\
X_{21} =& \Big[\frac{B(x)}{(b-1)x}, (1-b)y, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l(y)\Big] \\
X_{22} =& \Big[(1-b)x, \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l(y)\Big],
\end{aligned}
$$

are admissible.

$$
\boxed{X_{11} = \Big[\frac{(b-1)x}{B(x)}, \frac{A(y)}{-\mu y}, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l(y)\Big]}
$$

We have

$$
\begin{aligned}
&\partial_1^{\infty}(X_{11}) \subset \{t_3 = 1\}, \quad \partial_2^{0}(X_{11}) \subset \{t_5 = 1\}, \quad \partial_2^{\infty}(X_{11}) \subset \{t_5 = 1\}, \\
&\partial_3^{\infty}(X_{11}) \subset \{t_4 = 1\}, \quad \partial_4^{\infty}(X_{11}) \subset \{t_5 = 1\}, \quad \partial_5^{\infty}(X_{11}) \subset \{t_2 = 1\},
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_1^{0}(X_{11}) &= \Big[\frac{A(y)}{-\mu y}, \frac{1}{1-a}, \frac{y}{A(y)}, l(y)\Big] =: U^{(4)}, \\
\partial_3^{0}(X_{11}) &= \Big[\frac{b-1}{a\mu}, \frac{A(y)}{-\mu y}, \frac{aby+1}{abA(y)}, l(y)\Big] =: P', \\
\partial_4^{0}(X_{11}) &= \Big[\frac{(b-1)y}{B(y)}, \frac{A(y)}{-\mu y}, \frac{aby+1}{aA(y)}, l(y)\Big] =: Q', \\
\partial_5^{0}(X_{11}) &= \Big[\frac{(b-1)x}{B(x)}, \frac{A(c)}{-c\mu}, \frac{abx+1}{aA(x)}, \frac{c-x}{A(c)}\Big] \\
&\quad + \Big[\frac{(b-1)x}{B(x)}, \frac{A(y_2)}{-y_2\mu}, \frac{abx+1}{aA(x)}, \frac{y_2-x}{A(y_2)}\Big] =: M_1 + M_2.
\end{aligned}
$$

All these cycles are admissible by arguments similarly to those for U, P, and Q. For M_i $(i = 1, 2)$ we can see that the coordinate functions have distinct zeros and poles.

$$
\boxed{X_{12} = \Big[\frac{A(x)}{-\mu x}, \frac{(b-1)y}{B(y)}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l(y)\Big]}
$$

We have

$$
\begin{aligned}
&\partial_1^{0}(X_{12}) \subset \{t_4 = 1\}, \quad \partial_2^{0}(X_{12}) \subset \{t_5 = 1\}, \quad \partial_2^{\infty}(X_{12}) \subset \{t_4 = 1\}, \\
&\partial_3^{\infty}(X_{12}) \subset \{t_4 = 1\}, \quad \partial_4^{\infty}(X_{12}) \subset \{t_5 = 1\} \cup \{t_3 = 1\}, \quad \partial_5^{\infty}(X_{12}) \subset \{t_4 = 1\},
\end{aligned}
$$

and

$$\begin{aligned}
\partial_1^\infty(X_{12}) &= \Big[\frac{(b-1)y}{B(y)}, \frac{1}{1-a}, \frac{-\mu y}{A(y)}, l(y)\Big] =: Q'',\\
\partial_3^0(X_{12}) &= \Big[a, \frac{(b-1)y}{B(y)}, \frac{aby+1}{aA(y)}, l(y)\Big] =: Q''',\\
\partial_4^0(X_{12}) &= \Big[\frac{A(y)}{-\mu y}, \frac{(b-1)y}{B(y)}, \frac{aby+1}{aA(y)}, l(y)\Big] = -Q',\\
\partial_5^0(X_{12}) &= \Big[\frac{A(x)}{-\mu x}, \frac{(b-1)c}{B(c)}, \frac{abx+1}{aA(x)}, \frac{\mu(x-c)}{A(c)B(x)}\Big]\\
&\quad + \Big[\frac{A(x)}{-\mu x}, \frac{(b-1)y_2}{B(y_2)}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y_2)}{A(y_2)B(x)}\Big] =: N_1 + N_2.
\end{aligned}$$

Both Q'' and Q''' are admissible by argument similarly to that for Q. For N_i $(i = 1, 2)$ we can see that the coordinate functions have distinct zeros and poles except when $A(x) = 0$ which implies that $t_4 = 1$.

$$\boxed{X_{21} = \Big[\frac{B(x)}{(b-1)x}, (1-b)y, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l(y)\Big]}$$

We have

$$\begin{aligned}
&\partial_1^0(X_{21}) \subset \{t_3 = 1\}, \quad \partial_2^0(X_{21}) \subset \{t_5 = 1\}, \quad \partial_2^\infty(X_{21}) \subset \{t_4 = 1\},\\
&\partial_3^\infty(X_{21}) \subset \{t_4 = 1\}, \quad \partial_4^\infty(X_{21}) \subset \{t_5 = 1\}, \quad \partial_5^\infty(X_{21}) \subset \{t_2 = 1\},
\end{aligned}$$

and

$$\begin{aligned}
\partial_1^\infty(X_{21}) &= \Big[(1-b)y, \frac{1}{1-a}, \frac{y}{A(y)}, l(y)\Big] =: U^{(5)},\\
\partial_3^0(X_{21}) &= \Big[\frac{a\mu}{b-1}, (1-b)y, \frac{aby+1}{abA(y)}, l(y)\Big] =: P'',\\
\partial_4^0(X_{21}) &= \Big[\frac{B(y)}{(b-1)y}, (1-b)y, \frac{aby+1}{aA(y)}, l(y)\Big] =: Q^{(4)},\\
\partial_5^0(X_{21}) &= \Big[\frac{B(x)}{(b-1)x}, (1-b)c, \frac{abx+1}{aA(x)}, \frac{c-x}{A(c)}\Big]\\
&\quad + \Big[\frac{B(x)}{(b-1)x}, (1-b)y_2, \frac{abx+1}{aA(x)}, \frac{y_2-x}{A(y_2)}\Big] =: O_1 + O_2.
\end{aligned}$$

All these cycles are admissible by arguments similarly to those for U, P, and Q. For O_i $(i = 1, 2)$ we can see that the coordinate functions have distinct zeros and poles.

$$\boxed{X_{22} = \Big[(1-b)x, \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l(y)\Big]}$$

We have

$$\begin{aligned}
&\partial_2^0(X_{22}) \subset \{t_4 = 1\}, \quad \partial_2^\infty(X_{22}) \subset \{t_5 = 1\}, \quad \partial_3^\infty(X_{22}) \subset \{t_4 = 1\},\\
&\partial_4^\infty(X_{22}) \subset \{t_5 = 1\} \cup \{t_3 = 1\}, \qquad \partial_5^\infty(X_{22}) \subset \{t_4 = 1\},
\end{aligned}$$

and

$$\partial_1^0(X_{22}) = \Big[\frac{B(y)}{(b-1)y}, \frac{1}{1-a}, \frac{-\mu y}{A(y)}, l(y)\Big] = -Q'',$$
$$\partial_1^\infty(X_{22}) = \Big[\frac{B(y)}{(b-1)y}, b, \frac{\mu}{(b-1)A(y)}, l(y)\Big] =: Q^{(4)},$$
$$\partial_3^0(X_{22}) = \Big[\frac{b-1}{ab}, \frac{B(y)}{(b-1)y}, \frac{aby+1}{aA(y)}, l(y)\Big] = -Q''',$$
$$\partial_4^0(X_{22}) = \Big[(1-b)y, \frac{B(y)}{(b-1)y}, \frac{aby+1}{aA(y)}, l(y)\Big] =: Q^{(5)},$$
$$\begin{aligned}\partial_5^0(X_{22}) = {} & \Big[(1-b)x, \frac{B(c)}{(b-1)c}, \frac{abx+1}{aA(x)}, \frac{\mu(x-c)}{A(c)B(x)}\Big] \\ & + \Big[(1-b)x, \frac{B(y_2)}{(b-1)y_2}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y_2)}{A(y_2)B(x)}\Big] =: P_1 + P_2.\end{aligned}$$

All these cycles are admissible by argument similarly to that for Q. For P_i $(i = 1, 2)$ we can consider the coordinate functions and see that they have all distinct zeros and poles.

Step (5). Computation of X_1.

Set

$$\tilde{Z}(f_1, f_2) = \Big[f_1, f_2, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l(y)\Big].$$

We want to show that by throwing away the appropriate admissible and negligible cycle we have

$$X_1 = \tilde{Z}\Big(\frac{(b-1)f}{B}, \frac{A}{-\mu f}\Big) + \tilde{Z}\Big(\frac{A}{-\mu f}, \frac{(b-1)f}{B}\Big).$$

For this step we need to use Lemma 3.1(i) to get

$$\begin{aligned}X_{13} :=& \Big[\frac{(b-1)x}{B(x)}, \frac{A(y)}{-\mu y}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l(y)\Big] \\ =& \Big[\frac{(b-1)x}{B(x)}, \frac{A(y)}{-\mu y}, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l(y)\Big] \\ +& \Big[\frac{(b-1)x}{B(x)}, \frac{A(y)}{-\mu y}, \frac{abx+1}{aA(x)}, \frac{-\mu}{B(x)}, l(y)\Big] =: X_{11} + X_{14}.\end{aligned}$$

So it suffices to show that both X_{13} and X_{14} are admissible. It's obvious that X_{14} is the product of two admissible cycles

$$\Big[\frac{(b-1)x}{B(x)}, \frac{abx+1}{aA(x)}, \frac{-\mu}{B(x)}\Big], \quad \Big[\frac{A(y)}{-\mu y}, l(y)\Big]$$

while the admissibility of X_{13} can be shown by the same argument as that for X_{11} except for the last step

$$\begin{aligned}\partial_5^0(X_{13}) = {} & \Big[\frac{(b-1)x}{B(x)}, \frac{aA(c)}{(ab-b+1)c}, \frac{abx+1}{aA(x)}, \frac{\mu(x-c)}{A(c)B(x)}\Big] \\ & + \Big[\frac{(b-1)x}{B(x)}, \frac{aA(y_2)}{(ab-b+1)y_2}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y_2)}{A(y_2)B(x)}\Big] =: R_1 + R_2.\end{aligned}$$

By consideration of the zeros and poles of the coordinate functions we can show that R_i ($i = 1, 2$) is admissible because $B(x) = 0$ implies $t_3 = 1$.

Next we want to show

$$Z_3(F, F) = \tilde{Z}(F, F) \text{ for } F = \frac{A}{f},\ \frac{f}{B},\ \frac{A}{B}$$

where

$$Z_3\Big(\frac{f}{B}, \frac{f}{B}\Big) := \Big[\frac{(b-1)x}{B(x)}, \frac{(1-b)y}{B(y)}, \frac{abx+1}{aA(x)}, \frac{y-x}{yB(x)}, l(y)\Big], \tag{17}$$

$$Z_3\Big(\frac{A}{f}, \frac{A}{f}\Big) := \Big[\frac{A(x)}{-\mu x}, \frac{A(y)}{-\mu y}, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l(y)\Big], \tag{18}$$

$$Z_3\Big(\frac{A}{B}, \frac{A}{B}\Big) := \Big[\frac{A(x)}{B(x)}, \frac{A(y)}{B(y)}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l(y)\Big]. \tag{19}$$

To prove these it suffices to show the following: First,

$$\Big[\frac{(b-1)x}{B(x)}, \frac{(1-b)y}{B(y)}, \frac{abx+1}{aA(x)}, \frac{-\mu y}{A(y)}, l(y)\Big] \in C^1(F, 2) \wedge C^2(F, 3),$$
$$\Big[\frac{A(x)}{-\mu x}, \frac{A(y)}{-\mu y}, \frac{abx+1}{aA(x)}, \frac{-a\mu}{B(x)}, l(y)\Big] \in C^1(F, 2) \wedge C^2(F, 3)$$

are both admissible and negligible which is not hard to see. Then all the following are admissible and negligible:

$$\Big[b-1, \frac{(1-b)y}{B(y)}, \frac{abx+1}{aA(x)}, \frac{y-x}{yB(x)}, l(y)\Big] \in C^1(F, 1) \wedge C^2(F, 4),$$
$$\Big[\frac{(b-1)x}{B(x)}, b-1, \frac{abx+1}{aA(x)}, \frac{y-x}{yB(x)}, l(y)\Big] \in C^1(F, 1) \wedge C^2(F, 4),$$
$$\Big[\frac{-1}{\mu}, \frac{A(y)}{-\mu y}, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l(y)\Big] \in C^1(F, 1) \wedge C^2(F, 4),$$
$$\Big[\frac{A(x)}{-\mu x}, \frac{-1}{\mu}, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l(y)\Big] \in C^1(F, 1) \wedge C^2(F, 4),$$
$$\Big[\mu, \mu, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l(y)\Big] \in C^1(F, 1) \wedge C^1(F, 1) \wedge C^1(F, 3) \text{ for any } \mu \neq 0.$$

We only need to note that $B(x) = 0$ implies $t_3 = 1$ and that $yA(y) = 0$ implies $t_5 = 1$ for all the above cycles, and $B(y) = 0$ implies that $t_4 = 1$ for the first two cycles.

Step (6). Decomposition of $X_2 = Y_1 + Y_2 + Y_3 + Y_4$.

Put

$$v(x) = \frac{abx+1}{aA(x)}, \quad l_1(y) = 1 - \frac{y}{c}, \quad l_2(y) = \frac{y_2 - y}{y_2 B(y)},$$

which satisfies

$$l_1(y) l_2(y) = l(y) = 1 - \frac{k(c)}{k(y)}, \quad l_1(0) = l_2(0) = 1.$$

Then it follows from Lemma 3.2(i) that

$$X_2 = Y_1 + Y_2 + Y_3 + Y_4 \tag{20}$$

where all of the cycles

$$\begin{aligned}
Y_1 =& \Big[\frac{B(x)}{(b-1)x}, (b-1)y, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l_1(y)\Big],\\
Y_2 =& \Big[(b-1)x, \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l_1(y)\Big],\\
Y_3 =& \Big[\frac{(b-1)x}{B(x)}, (b-1)y, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l_2(y)\Big],\\
Y_4 =& \Big[(b-1)x, \frac{(b-1)y}{B(y)}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l_2(y)\Big]
\end{aligned}$$

are admissible. This breakup is the key step in the whole paper.

$$\boxed{Y_1 = \Big[\frac{B(x)}{(b-1)x}, (b-1)y, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l_1(y)\Big],}$$

We have

$$\begin{aligned}
&\partial_1^0(Y_1) \subset \{t_3 = 1\}, \quad \partial_2^0(Y_1) \subset \{t_5 = 1\}, \quad \partial_2^\infty(Y_1) \subset \{t_4 = 1\},\\
&\partial_3^\infty(Y_1) \subset \{t_4 = 1\}, \quad \partial_5^\infty(Y_1) \subset \{t_4 = 1\},
\end{aligned}$$

and

$$\begin{aligned}
\partial_1^\infty(Y_1) &= \Big[(b-1)y, \frac{1}{1-a}, \frac{y}{A(y)}, 1-\frac{y}{c}\Big],\\
\partial_3^0(Y_1) &= \Big[\frac{a\mu}{b-1}, (b-1)y, \frac{aby+1}{abA(y)}, 1-\frac{y}{c}\Big],\\
\partial_4^0(Y_1) &= \Big[\frac{B(y)}{(b-1)y}, (b-1)y, \frac{aby+1}{aA(y)}, 1-\frac{y}{c}\Big],\\
\partial_4^\infty(Y_1) &= \Big[\frac{B(x)}{(b-1)x}, \frac{(b-1)(a-1)}{a}, \frac{abx+1}{aA(x)}, \frac{ac-a+1}{ac}\Big],\\
\partial_5^0(Y_1) &= \Big[\frac{B(x)}{(b-1)x}, (b-1)c, \frac{abx+1}{aA(x)}, \frac{c-x}{aA(c)}\Big].
\end{aligned}$$

All these cycles are clearly admissible.

$$\boxed{Y_2 = \Big[(b-1)x, \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l_1(y)\Big]}$$

We have

$$\partial_2^0(Y_2) \subset \{t_4 = 1\}, \quad \partial_2^\infty(Y_2) \subset \{t_5 = 1\}, \quad \partial_3^\infty(Y_2) \subset \{t_4 = 1\}, \quad \partial_5^\infty(Y_2) \subset \{t_2 = 1\},$$

and

$$\begin{aligned}
\partial_1^0(Y_2) &= \Big[\frac{B(y)}{(b-1)y}, \frac{1}{1-a}, \frac{-\mu y}{A(y)}, 1-\frac{y}{c}\Big],\\
\partial_1^\infty(Y_2) &= \Big[\frac{B(y)}{(b-1)y}, b, \frac{\mu}{(b-1)A(y)}, 1-\frac{y}{c}\Big],\\
\partial_3^0(Y_2) &= \Big[\frac{1-b}{ab}, \frac{B(y)}{(b-1)y}, \frac{aby+1}{aA(y)}, 1-\frac{y}{c}\Big],\\
\partial_4^0(Y_2) &= \Big[(b-1)y, \frac{B(y)}{(b-1)y}, \frac{aby+1}{aA(y)}, \frac{c-y}{c}\Big],\\
\partial_4^\infty(Y_2) &= \Big[(b-1)x, \frac{ab-b+1}{(b-1)(a-1)}, \frac{abx+1}{aA(x)}, \frac{ac-a+1}{ac}\Big],\\
\partial_5^0(Y_2) &= \Big[(b-1)x, \frac{B(c)}{(b-1)c}, \frac{abx+1}{aA(x)}, \frac{\mu(x-c)}{A(c)B(x)}\Big].
\end{aligned}$$

All these cycles are clearly admissible.

$$\boxed{Y_3 = \Big[\frac{(b-1)x}{B(x)}, (b-1)y, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l_2(y)\Big],}$$

We have

$$\begin{aligned}
&\partial_1^\infty(Y_3) \subset \{t_3 = 1\}, \quad \partial_2^0(Y_3) \subset \{t_5 = 1\}, \quad \partial_2^\infty(Y_3) \subset \{t_4 = 1\},\\
&\partial_3^\infty(Y_3) \subset \{t_4 = 1\}, \quad \partial_5^\infty(Y_3) \subset \{t_4 = 1\},
\end{aligned}$$

and

$$\begin{aligned}
\partial_1^0(Y_3) &= \Big[(b-1)y, \frac{1}{1-a}, \frac{y}{A(y)}, \frac{y_2-y}{y_2B(y)}\Big],\\
\partial_3^0(Y_3) &= \Big[\frac{b-1}{a\mu}, (b-1)y, \frac{aby+1}{abA(y)}, \frac{y_2-y}{y_2B(y)}\Big],\\
\partial_4^0(Y_3) &= \Big[\frac{(b-1)y}{B(y)}, (b-1)y, \frac{aby+1}{aA(y)}, \frac{y_2-y}{y_2B(y)}\Big],\\
\partial_4^\infty(Y_3) &= \Big[\frac{(b-1)x}{B(x)}, \frac{(b-1)(a-1)}{a}, \frac{abx+1}{aA(x)}, \frac{ac}{ac-a+1}\Big],\\
\partial_5^0(Y_3) &= \Big[\frac{(b-1)x}{B(x)}, (b-1)y_2, \frac{abx+1}{aA(x)}, \frac{y_2-x}{A(y_2)}\Big].
\end{aligned}$$

All these cycles are clearly admissible.

$$\boxed{Y_4 = \Big[(b-1)x, \frac{(b-1)y}{B(y)}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l_2(y)\Big]}$$

We have

$$\partial_2^0(Y_4) \subset \{t_5 = 1\}, \quad \partial_2^\infty(Y_4) \subset \{t_4 = 1\}, \quad \partial_3^\infty(Y_4) \subset \{t_4 = 1\}, \quad \partial_5^\infty(Y_4) \subset \{t_2 = 1\},$$

and

$$\begin{aligned}
\partial_1^0(Y_4) &= \Big[\frac{(b-1)y}{B(y)}, \frac{1}{1-a}, \frac{-\mu y}{A(y)}, \frac{y_2-y}{y_2B(y)}\Big],\\
\partial_1^\infty(Y_4) &= \Big[\frac{(b-1)y}{B(y)}, b, \frac{\mu}{(b-1)A(y)}, \frac{y_2-y}{y_2B(y)}\Big],\\
\partial_3^0(Y_4) &= \Big[\frac{1-b}{ab}, \frac{(b-1)y}{B(y)}, \frac{aby+1}{aA(y)}, \frac{y_2-y}{y_2B(y)}\Big],\\
\partial_4^0(Y_4) &= \Big[(b-1)y, \frac{(b-1)y}{B(y)}, \frac{aby+1}{aA(y)}, \frac{y_2-y}{y_2B(y)}\Big],\\
\partial_4^\infty(Y_4) &= \Big[(b-1)x, \frac{(b-1)(a-1)}{ab-b+1}, \frac{abx+1}{aA(x)}, \frac{ac}{ac-a+1}\Big],\\
\partial_5^0(Y_4) &= \Big[(b-1)x, \frac{(b-1)y_2}{B(y_2)}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y_2)}{A(y_2)B(x)}\Big].
\end{aligned}$$

All these cycles are clearly admissible.

Step (7). Computation of Y_1+Y_2.

Set

$$\alpha = \frac{bc-c}{bc-c+1}, \qquad \delta = \frac{1}{b},$$

and

$$\begin{aligned}
v(x) &= \frac{abx+1}{aA(x)}, & g(x) &= \frac{B(x)}{(b-1)x}, & h(x) &= (b-1)x,\\
p_4(x,y) &= \frac{\mu(x-y)}{A(y)B(x)}, & q_4(x,y) &= \frac{y-x}{A(y)}, & s_4(x,y) &= \frac{(b-1)(y-x)}{B(y)},\\
r_4(x,y) &= \frac{(b-1)(y-x)}{xB(y)}, & w_4(x,y) &= \frac{y-x}{B(x)(y-1)}.
\end{aligned}$$

such that $\alpha l_1\big(1/(1-b)\big) = \delta v(\infty) = 1$. By Lemma 3.1(ii)(1) we get

$$\begin{aligned}
2[gh, gh, \delta v, q_4, \alpha l_1] =&[gh, gh, \delta v, q_4, \alpha l_1] + [gh, gh, \delta v, s_4, \alpha l_1]\\
=&[g, gh, \delta v, q_4, \alpha l_1] + [h, gh, \delta v, q_4, \alpha l_1]\\
&+[gh, g, \delta v, s_4, \alpha l_1] + [gh, h, \delta v, s_4, \alpha l_1]
\end{aligned}$$

are all admissible. Then applying Lemma 3.1 and Lemma 3.2 repeatedly we get

$$\begin{aligned}
&[g, gh, \delta v, q_4, \alpha l_1] + [gh, g, \delta v, s_4, \alpha l_1]\\
=&[g, gh, \delta v, q_4, \alpha l_1] + [gh, g, \delta v, r_4, \alpha l_1]\\
=&[g, gh, v, q_4, \alpha l_1] + [gh, g, v, r_4, \alpha l_1]\\
=&[g, gh, v, q_4, \alpha l_1] + [gh, g, v, w_4, \alpha l_1]\\
=&[g, gh, v, q_4, l_1] + [gh, g, v, w_4, l_1]\\
=&[g, gh, v, q_4, l_1] + [gh, g, v, p_4, l_1]\\
=&[g, h, v, q_4, l_1] + [h, g, v, p_4, l_1] + [g, g, v, q_4, l_1] + [g, g, v, p_4, l_1]\\
=&[g, h, v, q_4, l_1] + [h, g, v, p_4, l_1] + 2[g, g, v, p_4, l_1].
\end{aligned}$$

Again applying Lemma 3.2(i) and (ii) and Lemma 3.1(ii), we have

$$\begin{aligned}
&[h, gh, \delta v, q_4, \alpha l_1] + [gh, h, \delta v, s_4, \alpha l_1] \\
=&[h, gh, \delta v, q_4, l_1] + [gh, h, \delta v, s_4, l_1] \\
=&[h, gh, \delta v, q_4, l_1] + [gh, h, \delta v, q_4, l_1] \\
=&[h, g, \delta v, q_4, l_1] + [g, h, \delta v, q_4, l_1] + 2[h, h, \delta v, q_4, l_1] \\
=&[h, g, v, q_4, l_1] + [g, h, v, q_4, l_1] + 2[h, h, \delta v, q_4, l_1] \\
=&[h, g, v, p_4, l_1] + [g, h, v, q_4, l_1] + 2[h, h, \delta v, q_4, l_1].
\end{aligned}$$

Let's prove the admissibility of all the cycles appearing in these equations.

- $[gh, gh, \delta v, q_4, \alpha l_1]$ is admissible.

$$\boxed{[B, B] := [gh, gh, \delta v, q_4, \alpha l_1] = \Big[B(x), B(y), \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, \alpha l_1(y)\Big]}$$

We have

$$\begin{aligned}
&\partial_1^\infty[B, B] \subset \{t_3 = 1\}, \quad \partial_2^0[B, B] \subset \{t_5 = 1\}, \quad \partial_2^\infty[B, B] \subset \{t_4 = 1\}, \\
&\partial_3^\infty[B, B] \subset \{t_4 = 1\}, \quad \partial_5^\infty[B, B] \subset \{t_4 = 1\},
\end{aligned}$$

and

$$\begin{aligned}
\partial_1^0[B, B] &= \Big[B(y), \frac{1}{b}, \frac{B(y)}{(b-1)A(y)}, \alpha l_1(y)\Big], \\
\partial_3^0[B, B] &= \Big[\frac{\mu}{b}, B(y), \frac{aby+1}{abA(y)}, \alpha l_1(y)\Big], \\
\partial_4^0[B, B] &= \Big[B(y), B(y), \frac{aby+1}{abA(y)}, \alpha l_1(y)\Big], \\
\partial_4^\infty[B, B] &= \Big[B(x), \mu, \frac{abx+1}{abA(x)}, \frac{(b-1)(ac-a+1)}{a(bc-c+1)}\Big], \\
\partial_5^0[B, B] &= \Big[B(x), B(c), \frac{abx+1}{abA(x)}, \frac{c-x}{A(c)}\Big].
\end{aligned}$$

All these cycles are clearly admissible by our choice of α and δ.

- $[g, gh, \cdots]$ are admissible.

$$\boxed{[B/f, B] := [g, gh, \delta v, q_4, \alpha l_1] = \Big[\frac{B(x)}{(b-1)x}, B(y), \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, \alpha l_1(y)\Big]}$$

We have

$$\begin{aligned}
&\partial_2^0[B/f, B] \subset \{t_5 = 1\}, \quad \partial_2^\infty[B/f, B] \subset \{t_4 = 1\}, \\
&\partial_3^\infty[B/f, B] \subset \{t_4 = 1\}, \quad \partial_5^\infty[B/f, B] \subset \{t_4 = 1\},
\end{aligned}$$

and

$$\begin{aligned}
\partial_1^0[B/f,B] &= \Big[B(y), \frac{1}{b}, \frac{B(y)}{(b-1)A(y)}, \alpha l_1(y)\Big],\\
\partial_1^\infty[B/f,B] &= \Big[B(y), \frac{1}{b(1-a)}, \frac{y}{A(y)}, \alpha l_1(y)\Big],\\
\partial_3^0[B/f,B] &= \Big[\frac{a\mu}{b-1}, B(y), \frac{aby+1}{abA(y)}, \alpha l_1(y)\Big],\\
\partial_4^0[B/f,B] &= \Big[\frac{B(y)}{(b-1)y}, B(y), \frac{aby+1}{abA(y)}, \alpha l_1(y)\Big],\\
\partial_4^\infty[B/f,B] &= \Big[\frac{B(x)}{(b-1)x}, \mu, \frac{abx+1}{abA(x)}, \frac{(b-1)(ac-a+1)}{a(bc-c+1)}\Big],\\
\partial_5^0[B/f,B] &= \Big[\frac{B(x)}{(b-1)x}, B(c), \frac{abx+1}{abA(x)}, \frac{c-x}{A(c)}\Big].
\end{aligned}$$

All these cycles are clearly admissible by our choice of α. Then by similar argument we can see that

$$[g, gh, v, q_4, \alpha l_1], \quad \text{and} \quad [g, gh, \delta, q_4, \alpha l_1]$$

are both admissible because we didn't use $t_3 = 1$ in the above.

$$\boxed{[B/f,B]_1 := [g, gh, v, q_4, l_1] = \Big[\frac{B(x)}{(b-1)x}, B(y), \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l_1(y)\Big]}$$

The same proof as above except whenever we used $\alpha l_1(1/(1-b)) = 1$ (namely $t_5 = 1$) before we have to use $v(1/(1-b)) = 1$ now.

- $[gh, g, \cdots]$ are admissible.

$$\boxed{[B,B/f] := [gh, g, \delta v, s_4, \alpha l_1] = \Big[B(x), \frac{B(y)}{(b-1)y}, \frac{abx+1}{abA(x)}, \frac{(b-1)(y-x)}{B(y)}, \alpha l_1(y)\Big]}$$

We have

$$\begin{aligned}
&\partial_1^0[B,B/f] \subset \{t_4 = 1\}, \quad \partial_1^\infty[B,B/f] \subset \{t_3 = 1\}, \quad \partial_2^0[B,B/f] \subset \{t_5 = 1\},\\
&\partial_4^\infty[B,B/f] \subset \{t_5 = 1\}, \quad \partial_5^\infty[B,B/f] \subset \{t_2 = 1\},
\end{aligned}$$

and

$$\begin{aligned}
\partial_2^\infty[B,B/f] &= \Big[B(x), \frac{abx+1}{abA(x)}, (1-b)x, \alpha\Big],\\
\partial_3^0[B,B/f] &= \Big[\frac{\mu}{b}, \frac{B(y)}{(b-1)y}, \frac{(b-1)(aby+1)}{abB(y)}, \alpha l_1(y)\Big],\\
\partial_3^\infty[B,B/f] &= \Big[-\mu, \frac{B(y)}{(b-1)y}, \frac{(b-1)A(y)}{B(y)}, \alpha l_1(y)\Big],\\
\partial_4^0[B,B/f] &= \Big[B(y), \frac{B(y)}{(b-1)y}, \frac{aby+1}{abA(y)}, \alpha l_1(y)\Big],\\
\partial_5^0[B,B/f] &= \Big[B(x), \frac{B(c)}{(b-1)c}, \frac{abx+1}{abA(x)}, \frac{(b-1)(c-x)}{B(c)}\Big].
\end{aligned}$$

All these cycles are clearly admissible by our choice of α.

$$\boxed{[B,B/f]' := [gh, g, \delta v, s_4/r_4, \alpha l_1] = \Big[B(x), \frac{B(y)}{(b-1)y}, \frac{abx+1}{abA(x)}, x, \alpha l_1(y)\Big]}$$

This a product of two admissible cycles.

$$\boxed{[B,B/f]_1 := [gh, g, \delta v, r_4, \alpha l_1] = \Big[B(x), \frac{B(y)}{(b-1)y}, \frac{abx+1}{abA(x)}, \frac{(b-1)(y-x)}{xB(y)}, \alpha l_1(y)\Big]}$$

The same proof above works if we notice now that

$$\partial_1^0[B,B/f] = \Big[\frac{B(y)}{(b-1)y}, \frac{1}{b}, b-1, \alpha l_1(y)\Big]$$

is admissible.

$$\boxed{[B,B/f]_2 := [gh, g, v, r_4, \alpha l_1] = \Big[B(x), \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{(b-1)(y-x)}{xB(y)}, \alpha l_1(y)\Big]}$$

We have

$$\begin{aligned} \partial_1^0[B,B/f]_2 \subset \{t_3 = 1\}, &\quad \partial_2^0[B,B/f]_2 \subset \{t_5 = 1\}, \\ \partial_4^\infty[B,B/f]_2 \subset \{t_5 = 1\}, &\quad \partial_5^\infty[B,B/f]_2 \subset \{t_2 = 1\}, \end{aligned}$$

and

$$\begin{aligned} \partial_1^\infty[B,B/f]_2 &= \Big[\frac{B(y)}{(b-1)y}, b, \frac{1-b}{B(y)}, \alpha l_1(y)\Big], \\ \partial_2^\infty[B,B/f]_2 &= \Big[B(x), \frac{abx+1}{aA(x)}, 1-b, \alpha\Big], \\ \partial_3^0[B,B/f]_2 &= \Big[\frac{\mu}{b}, \frac{B(y)}{(b-1)y}, \frac{(1-b)(aby+1)}{B(y)}, \alpha l_1(y)\Big], \\ \partial_3^\infty[B,B/f]_2 &= \Big[-\mu, \frac{B(y)}{(b-1)y}, \frac{a(b-1)A(y)}{(a-1)B(y)}, \alpha l_1(y)\Big], \\ \partial_4^0[B,B/f]_2 &= \Big[B(y), \frac{B(y)}{(b-1)y}, \frac{aby+1}{aA(y)}, \alpha l_1(y)\Big], \\ \partial_5^0[B,B/f]_2 &= \Big[B(x), \frac{B(c)}{(b-1)c}, \frac{abx+1}{abA(x)}, \frac{(b-1)(c-x)}{xB(c)}\Big]. \end{aligned}$$

All these cycles are clearly admissible by our choice of α. The same proof shows that $[gh, g, \delta, r_4, \alpha l_1]$ is admissible.

$$\boxed{[B,B/f]_2' := [gh, g, v, r_4/w_4, \alpha l_1] = \Big[B(x), \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{(b-1)B(x)(y-1)}{xB(y)}, \alpha l_1(y)\Big]}$$

This is a sum of two admissible cycles:

$$\Big[B(x), \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{(y-1)}{B(y)}, \alpha l_1(y)\Big] + \Big[B(x), \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{(b-1)B(x)}{x}, \alpha l_1(y)\Big].$$

$$\boxed{[B, B/f]_3 := [gh, g, v, w_4, \alpha l_1] = \Big[B(x), \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{y-x}{B(x)(y-1)}, \alpha l_1(y)\Big]}$$

We have

$$\partial_1^0[B, B/f]_3 \subset \{t_3 = 1\}, \quad \partial_2^0[B, B/f]_3 \subset \{t_5 = 1\}, \quad \partial_5^\infty[B, B/f]_3 \subset \{t_2 = 1\},$$

and

$$\begin{aligned}
\partial_1^\infty[B, B/f]_3 &= \Big[\frac{B(y)}{(b-1)y}, b, \frac{y}{(1-b)(y-1)}, \alpha l_1(y)\Big],\\
\partial_2^\infty[B, B/f]_3 &= \Big[B(x), \frac{abx+1}{abA(x)}, \frac{x}{B(x)}, \alpha\Big],\\
\partial_3^0[B, B/f]_3 &= \Big[\frac{\mu}{b}, \frac{B(y)}{(b-1)y}, \frac{a(1-b)(aby+1)}{\mu(y-1)}, \alpha l_1(y)\Big],\\
\partial_3^\infty[B, B/f]_3 &= \Big[-\mu, \frac{B(y)}{(b-1)y}, \frac{A(y)}{\mu(1-y)}, \alpha l_1(y)\Big],\\
\partial_4^0[B, B/f]_3 &= \Big[B(y), \frac{B(y)}{(b-1)y}, \frac{aby+1}{aA(y)}, \alpha l_1(y)\Big],\\
\partial_4^\infty[B, B/f]_3 &== \Big[B(x), \frac{1}{b-1}, \frac{abx+1}{aA(x)}, \frac{\alpha(c-1)}{c}\Big]\\
\partial_5^0[B, B/f]_3 &= \Big[B(x), \frac{B(c)}{(b-1)c}, \frac{abx+1}{aA(x)}, \frac{c-x}{B(x)(c-1)}\Big].
\end{aligned}$$

All these cycles are clearly admissible.

$$\boxed{[B, B/f]_4 := [gh, g, v, w_4, l_1] = \Big[B(x), \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{y-x}{B(x)(y-1)}, l_1(y)\Big]}$$

Note that in the above proof for $[B, B/f]_3$ the choice of α is not essential because whenever $B(x) = 0$ we have $(abx+1)/aA(x) = 1$. The same reason shows that $[gh, g, v, w_4, \alpha]$ is admissible.

$$\boxed{[B, B/f]_4' := [gh, g, v, w_4/p_4, l_1] = \Big[B(x), \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{A(y)}{1-y}, l_1(y)\Big]}$$

This is a product of two admissible cycles.

$$\boxed{[B, B/f]_5 := [gh, g, v, p_4, l_1] = \Big[B(x), \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l_1(y)\Big]}$$

The proof for $[B, B/f]_4$ can be adapted here without any change.

- $[gh, h, \cdots]$ are admissible.

$$\boxed{[B,f] := [gh, h, \delta v, s_4, \alpha l_1] = \left[B(x), (b-1)y, \frac{abx+1}{abA(x)}, \frac{(b-1)(y-x)}{B(y)}, \alpha l_1(y)\right]}$$

We have

$$\partial_1^0[B,f] \subset \{t_4 = 1\}, \quad \partial_1^\infty[B,f] \subset \{t_3 = 1\}, \quad \partial_2^\infty[B,f] \subset \{t_4 = 1\},$$
$$\partial_4^\infty[B,f] \subset \{t_5 = 1\}, \quad \partial_5^\infty[B,f] \subset \{t_4 = 1\},$$

and

$$\begin{aligned}
\partial_2^0[B,f] &= \left[B(x), \frac{abx+1}{abA(x)}, (1-b)x, \alpha\right], \\
\partial_3^0[B,B/f] &= \left[\frac{\mu}{b}, (b-1)y, \frac{(b-1)(aby+1)}{abB(y)}, \alpha l_1(y)\right], \\
\partial_3^\infty[B,B/f] &= \left[-\mu, (b-1)y, \frac{(b-1)A(y)}{B(y)}, \alpha l_1(y)\right], \\
\partial_4^0[B,f] &= \left[B(y), (b-1)y, \frac{aby+1}{abA(y)}, \alpha l_1(y)\right], \\
\partial_5^0[B,f] &= \left[B(x), (1-b)c, \frac{abx+1}{abA(x)}, \frac{(b-1)(c-x)}{B(c)}\right].
\end{aligned}$$

All these cycles are clearly admissible. Note that we never use the property of α, (namely $t_5 = 1$), so the same proof shows that $[gh, h, \delta v, q_4, l_1]$ is admissible.

$$\boxed{[B,f]' := [gh, h, \delta v, s_4/q_4, l_1] = \left[B(x), (b-1)y, \frac{abx+1}{abA(x)}, \frac{(b-1)A(y)}{B(y)}, l_1(y)\right]}$$

This is a product of two admissible cycles.

$$\boxed{[B,f]_1 := [gh, h, \delta v, q_4, l_1] = \left[B(x), (b-1)y, \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, l_1(y)\right]}$$

The same proof for $[B,f]$ works.

- $[h, gh, \cdots]$ are admissible.

$$\boxed{[f,B] := [h, gh, \delta v, q_4, \alpha l_1] = \left[(b-1)x, B(y), \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, \alpha l_1(y)\right]}$$

We have

$$\partial_1^\infty[f,B] \subset \{t_3 = 1\}\Big], \quad \partial_2^0[f,B] \subset \{t_5 = 1\}, \quad \partial_2^\infty[f,B] \subset \{t_4 = 1\},$$
$$\partial_3^\infty[f,B] \subset \{t_4 = 1\}, \quad \partial_5^\infty[f,B] \subset \{t_4 = 1\},$$

and

$$\partial_1^0[f,B] = \Big[B(y), \frac{1}{b}, \frac{y}{A(y)}, \alpha l_1(y)\Big],$$
$$\partial_3^0[f,B] = \Big[\frac{1-b}{ab}, B(y), \frac{aby+1}{abA(y)}, \alpha l_1(y)\Big],$$
$$\partial_4^0[f,B] = \Big[(b-1)y, B(y), \frac{aby+1}{abA(y)}, \alpha l_1(y)\Big],$$
$$\partial_4^\infty[f,B] = \Big[(b-1)x, -\mu, \frac{abx+1}{abA(x)}, \frac{(b-1)(ac-a+1)}{a(bc-c+1)}\Big],$$
$$\partial_5^0[f,B] = \Big[(b-1)x, B(c), \frac{abx+1}{abA(x)}, \frac{c-x}{A(c)}\Big].$$

All these cycles are clearly admissible by our choice of δ.

$$\boxed{[f,B]_1 := [h, gh, \delta v, q_4, l_1] = \Big[(b-1)x, B(y), \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, l_1(y)\Big]}$$

Note that we use the property of α (namely $t_5 = 1$) only for $\partial_2^0[f,B]$ so the same proof applies because

$$\partial_2^0[f,B]_1 = \Big[(b-1)x, \frac{abx+1}{abA(x)}, \frac{B(x)}{-\mu}, \frac{1}{\alpha}\Big]$$

is clearly admissible.

- $[g, h, \cdots]$ are admissible.

$$\boxed{[B/f, f] := [g, h, \delta v, q_4, l_1] = \Big[\frac{B(x)}{(b-1)x}, (b-1)y, \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, l_1(y)\Big]}$$

We have

$$\partial_2^0[B/f,f] \subset \{t_5 = 1\}, \quad \partial_2^\infty[B/f,f] \subset \{t_4 = 1\},$$
$$\partial_3^\infty[B/f,f] \subset \{t_4 = 1\}, \quad \partial_5^\infty[B/f,f] \subset \{t_4 = 1\},$$

and

$$\partial_1^0[B/f,f] = \Big[(b-1)y, \frac{1}{b}, \frac{B(y)}{(b-1)A(y)}, l_1(y)\Big],$$
$$\partial_1^\infty[B/f,f] = \Big[(b-1)y, \frac{1}{b(1-a)}, \frac{y}{A(y)}, l_1(y)\Big],$$
$$\partial_3^0[B/f,f] = \Big[\frac{a\mu}{1-b}, (b-1)y, \frac{aby+1}{abA(y)}, l_1(y)\Big],$$
$$\partial_4^0[B/f,f] = \Big[\frac{B(y)}{(b-1)y}, (b-1)y, \frac{aby+1}{abA(y)}, l_1(y)\Big],$$
$$\partial_4^\infty[B/f,f] = \Big[\frac{B(x)}{(b-1)x}, \frac{(a-1)(b-1)}{a}, \frac{abx+1}{abA(x)}, \frac{ac-a+1}{ac}\Big],$$
$$\partial_5^0[B/f,f] = \Big[\frac{B(x)}{(b-1)x}, (1-b)c, \frac{abx+1}{abA(x)}, \frac{c-x}{A(c)}\Big].$$

All these cycles are clearly admissible. Note that we never use the property of δ, (namely $t_3 = 1$), so the same proof shows that $[g, h, v, q_4, l_1]$ and $[g, h, \delta, q_4, l_1]$ are admissible.

- $[h, g, \cdots]$ are admissible.

$$\boxed{[f, B/f] := [h, g, \delta v, q_4, l_1] = \Big[(b-1)x, \frac{B(y)}{(b-1)y}, \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, l_1(y)\Big]}$$

We have

$$\begin{aligned}
\partial_1^\infty[f, B/f] \subset \{t_3 = 1\}, \quad \partial_2^\infty[f, B/f] \subset \{t_5 = 1\}, \\
\partial_3^\infty[f, B/f] \subset \{t_4 = 1\}, \quad \partial_5^\infty[f, B/f] \subset \{t_2 = 1\},
\end{aligned}$$

and

$$\begin{aligned}
\partial_1^0[f, B/f] &= \Big[\frac{B(y)}{(b-1)y}, \frac{1}{b(1-a)}, \frac{y}{A(y)}, l_1(y)\Big], \\
\partial_2^0[f, B/f] &= \Big[(b-1)x, \frac{abx+1}{abA(x)}, \frac{B(x)}{-\mu}, \frac{1}{\alpha}\Big] \\
\partial_3^0[f, B/f] &= \Big[\frac{1-b}{ab}, \frac{B(y)}{(b-1)y}, \frac{aby+1}{abA(y)}, l_1(y)\Big], \\
\partial_4^0[f, B/f] &= \Big[(b-1)y, \frac{B(y)}{(b-1)y}, \frac{aby+1}{abA(y)}, l_1(y)\Big], \\
\partial_4^\infty[f, B/f] &= \Big[(b-1)x, \frac{ab-b+1}{(a-1)(b-1)}, \frac{abx+1}{abA(x)}, \frac{ac-a+1}{ac}\Big], \\
\partial_5^0[f, B/f] &= \Big[(b-1)x, \frac{B(c)}{(b-1)c}, \frac{abx+1}{abA(x)}, \frac{c-x}{A(c)}\Big].
\end{aligned}$$

All these cycles are clearly admissible.

$$\boxed{[f, B/f]' := [h, g, \delta v, p_4/q_4, l_1] = \Big[(b-1)x, \frac{B(y)}{(b-1)y}, \frac{abx+1}{abA(x)}, \frac{-\mu}{B(x)}, l_1(y)\Big]}$$

In the above proof the only place we use $q_4 = 1$ (namely $t_4 = 1$) is for $\partial_3^\infty[f, B/f]$. However, it's still true in $[f, B/f]_1'$ that $t_4 = 1$ if $A(x) = 0$. Now the only things that need checking are

$$\partial_4^0[f, B/f]' \subset \{t_3 = 1\}, \quad \partial_4^\infty[f, B/f]' = \Big[-1, \frac{B(y)}{(b-1)y}, \frac{1}{b}, l_1(y)\Big].$$

$$\boxed{[f, B/f]_1 := [h, g, \delta v, p_4, l_1] = \Big[(b-1)x, \frac{B(y)}{(b-1)y}, \frac{abx+1}{abA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l_1(y)\Big]}$$

Similar to $[f, B/f]'$ the only things that need checking are

$$\partial_4^0[f, B/f]_1 = \partial_4^0[f, B/f], \quad \partial_4^\infty[f, B/f]_1 = \partial_4^\infty[f, B/f] + \partial_4^\infty[f, B/f]_1'.$$

$$\boxed{[f,B/f]_2 := [h,g,v,p_4,l_1] = \Big[(b-1)x, \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l_1(y)\Big]}$$

In the above proofs for $[f,B/f]$ and $[f,B/f]_1$ the only place we use the property of δ (namely $t_3=1$) for $\partial_1^\infty[f,B/f]$. But

$$\partial_1^\infty[f,B/f]_2 = \Big[\frac{B(y)}{(b-1)y}, \frac{1}{b}, \frac{\mu}{(b-1)A(y)}, l_1(y)\Big]$$

which is admissible. This shows that $[f,B/f]_2$ is admissible.

- $[g,g,\cdots]$ are admissible.

$$\boxed{[B/f,B/f] := [g,g,v,q_4,l_1] = \Big[\frac{B(x)}{(b-1)x}, \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l_1(y)\Big]}$$

We have

$$\begin{aligned}
\partial_1^0[B/f,B/f] \subset \{t_3=1\}, \quad \partial_2^\infty[B/f,B/f] \subset \{t_5=1\},\\
\partial_3^\infty[B/f,B/f] \subset \{t_4=1\}, \quad \partial_5^\infty[B/f,B/f] \subset \{t_2=1\},
\end{aligned}$$

and

$$\begin{aligned}
\partial_1^\infty[B/f,B/f] &= \Big[\frac{B(y)}{(b-1)y}, \frac{1}{1-a}, \frac{y}{A(y)}, l_1(y)\Big],\\
\partial_2^0[B/f,B/f] &= \Big[\frac{B(x)}{(b-1)x}, \frac{abx+1}{aA(x)}, \frac{B(x)}{-\mu}, \frac{1}{\alpha}\Big]\\
\partial_3^0[B/f,B/f] &= \Big[\frac{a\mu}{b-1}, \frac{B(y)}{(b-1)y}, \frac{aby+1}{abA(y)}, l_1(y)\Big],\\
\partial_4^0[B/f,B/f] &= \Big[\frac{B(y)}{(b-1)y}, \frac{B(y)}{(b-1)y}, \frac{aby+1}{aA(y)}, l_1(y)\Big],\\
\partial_4^\infty[B/f,B/f] &= \Big[\frac{B(x)}{(b-1)x}, \frac{ab-b+1}{(a-1)(b-1)}, \frac{abx+1}{aA(x)}, \frac{ac-a+1}{ac}\Big],\\
\partial_5^0[B/f,B/f] &= \Big[\frac{B(x)}{(b-1)x}, \frac{B(c)}{(b-1)c}, \frac{abx+1}{aA(x)}, \frac{c-x}{A(c)}\Big].
\end{aligned}$$

All these cycles are clearly admissible.

$$\boxed{[B/f,B/f]_1 := [g,g,v,p_4,l_1] = \Big[\frac{B(x)}{(b-1)x}, \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l_1(y)\Big]}$$

In the above proof the only place we use $q_4=1$ (namely $t_4=1$) is for $\partial_3^\infty[B/f,B/f]$. However, it's still true in $[B/f,B/f]_1$ that $p_4=1$ if $A(x)=0$. Now the only things that need checking are

$$\begin{aligned}
\partial_4^0[B/f,B/f]_1 =& \partial_4^0[B/f,B/f],\\
\partial_4^\infty[B/f,B/f]_1 =& \partial_4^\infty[B/f,B/f] \quad \text{(if } B(x)=0 \text{ then } t_3=1).
\end{aligned}$$

$$\boxed{[B/f,B/f]_1' := [g,g,v,p_4/q_4,l_1] = \Big[\frac{B(x)}{(b-1)x},\frac{B(y)}{(b-1)y},\frac{abx+1}{aA(x)},\frac{-\mu}{B(x)},l_1(y)\Big]}$$

In the above proof the only place we use $q_4=1$ (namely $t_4=1$) is for $\partial_3^\infty[B/f,B/f]$. However, it's still true in $[B/f,B/f]_3$ that $t_4=1$ if $A(x)=0$. Now the only things that need checking are

$$\partial_4^0[B/f,B/f]_1' \subset \{t_1=1\},\quad \partial_4^\infty[B/f,B/f]_1' \subset \{t_3=1\}.$$

- $[h,h,\delta v,q_4,l_1]$ is admissible.

$$\boxed{[f,f] := [h,h,\delta v,q_4,l_1] = \Big[(b-1)x,(b-1)y,\frac{abx+1}{abA(x)},\frac{y-x}{A(y)},l_1(y)\Big]}$$

We have

$$\begin{aligned}&\partial_1^\infty[f,f]\subset\{t_3=1\},\quad \partial_2^0[f,f]\subset\{t_5=1\},\quad \partial_2^\infty[f,f]\subset\{t_4=1\},\\ &\partial_3^\infty[f,f]\subset\{t_4=1\},\quad \partial_5^\infty[f,f]\subset\{t_4=1\},\end{aligned}$$

and

$$\begin{aligned}\partial_1^0[f,f] &= \Big[(b-1)y,\frac{1}{b(1-a)},\frac{y}{A(y)},l_1(y)\Big],\\ \partial_3^0[f,f] &= \Big[\frac{1-b}{ab},(b-1)y,\frac{aby+1}{abA(y)},l_1(y)\Big],\\ \partial_4^0[f,f] &= \Big[(b-1)y,(b-1)y,\frac{aby+1}{abA(y)},l_1(y)\Big],\\ \partial_4^\infty[f,f] &= \Big[(b-1)x,\frac{(a-1)(b-1)}{a},\frac{abx+1}{abA(x)},\frac{ac-a+1}{ac}\Big],\\ \partial_5^0[f,f] &= \Big[(b-1)x,(1-b)c,\frac{abx+1}{abA(x)},\frac{c-x}{A(c)}\Big].\end{aligned}$$

All these cycles are clearly admissible. Note that we didn't use $t_1=1$ or $t_2=1$ in the above so the same argument implies that

$$[b-1,h,\delta v,q_4,l_1],\ [h,b-1,\delta v,q_4,l_1],\ [b-1,b-1,\delta v,q_4,l_1]$$

are all admissible. So we get

$$[h,h,\delta v,q_4,l_1] = \Big[x,y,\frac{abx+1}{abA(x)},\frac{y-x}{A(y)},l_1(y)\Big] \tag{21}$$

Step (8). Computation of Y_3+Y_4.

Claim 8.1. Under non-degeneracy assumption

$$Y_{31}' := \Big[\frac{(1-b)x}{B(x)},(1-b)y,\frac{abA(x)}{abx+1},\frac{ab(y-x)}{aby+1},l_2(y)\Big] = -Y_3'.$$

$$\boxed{Y'_{31} = \Big[\frac{(1-b)x}{B(x)}, (1-b)y, \frac{aA(x)}{abx+1}, \frac{ab(y-x)}{aby+1}, l_2(y)\Big]}$$

We have

$$\partial_1^\infty Y'_{31} \subset \{t_3 = 1\}, \quad \partial_2^0 Y'_{31} \subset \{t_5 = 1\}, \quad \partial_2^\infty Y'_{31} \subset \{t_4 = 1\},$$
$$\partial_3^\infty Y'_{31} \subset \{t_4 = 1\}, \quad \partial_5^\infty Y'_{31} \subset \{t_4 = 1\},$$

and

$$\partial_1^0 Y'_{31} = \Big[(1-b)y, 1-a, \frac{aby}{aby+1}, l_2(y)\Big],$$
$$\partial_3^0 Y'_{31} = \Big[\frac{(a-1)(1-b)}{ab-b+1}, (1-b)y, \frac{abA(y)}{aby+1}, l_2(y)\Big],$$
$$\partial_4^0 Y'_{31} = \Big[\frac{(1-b)y}{B(y)}, (1-b)y, \frac{aA(y)}{aby+1}, l_2(y)\Big]],$$
$$\partial_4^\infty Y'_{31} = \Big[\frac{(1-b)x}{B(x)}, \frac{b-1}{ab}, \frac{aA(x)}{abx+1}, \frac{ac-a}{ac-a+1}\Big],$$
$$\partial_5^0 Y'_{31} = \Big[\frac{(1-b)x}{B(x)}, (1-b)y_2, \frac{aA(x)}{abx+1}, \frac{ab(y_2-x)}{aby_2+1}\Big].$$

All these cycles are clearly admissible. For the last one, we need (13).

$$\boxed{Y''_{31} = \Big[\frac{(1-b)x}{B(x)}, (1-b)y, b, \frac{ab(y-x)}{aby+1}, l_2(y)\Big]}$$

In the above proof for Y'_{31} there is only one place where we used $t_3 = 1$, namely for ∂_1^∞. But

$$\partial_3^\infty Y''_{31} = \Big[(1-b)y, b, \frac{abB(y)}{(b-1)(aby+1)}, l_2(y)\Big]$$

which is admissible because if $B(y) = 0$ then $(1-b)y = 1$.

$$\boxed{Y'_{32} = \Big[\frac{(1-b)x}{B(x)}, (1-b)y, \frac{abA(x)}{abx+1}, \frac{aby+1}{abA(y)}, l_2(y)\Big]}$$

This is a product of two admissible cycles. From the above we get

$$Y'_{31} = Y'_{31} + Y'_{32} = Y'_{31} + Y'_{32} + Y''_{31} = -Y'_3.$$

Claim 8.1 is proved.

Claim 8.2. Under non-degeneracy assumption

$$Y'_{41} := \Big[(1-b)x, \frac{(1-b)y}{B(y)}, \frac{abA(x)}{abx+1}, \frac{(ab-b+1)(y-x)}{(aby+1)B(x)}, l_2(y)\Big] = -Y'_4.$$

$$\boxed{Y'_{41} = \Big[(1-b)x, \frac{B(y)}{(1-b)y}, \frac{aA(x)}{abx+1}, \frac{(ab-b+1)(y-x)}{(aby+1)B(x)}, l_2(y)\Big]}$$

We have

$$\partial_2^0 Y'_{41} \subset \{t_4 = 1\}, \quad \partial_2^\infty Y'_{41} \subset \{t_5 = 1\}, \quad \partial_5^\infty Y'_{41} \subset \{t_4 = 1\}, \quad \partial_3^\infty Y'_{41} \subset \{t_4 = 1\},$$

and

$$\begin{aligned}
\partial_1^\infty Y'_{41} &= \Big[\frac{B(y)}{(1-b)y}, 1-a, \frac{(ab-b+1)y}{aby+1}, l_2(y)\Big],\\
\partial_1^0 Y'_{41} &= \Big[\frac{B(y)}{(1-b)y}, \frac{1}{b}, \frac{ab-b+1}{(1-b)(aby+1)}, l_2(y)\Big],\\
\partial_3^0 Y'_{41} &= \Big[\frac{(a-1)(1-b)}{a}, \frac{B(y)}{(1-b)y}, \frac{aA(y)}{aby+1}, l_2(y)\Big],\\
\partial_4^0 Y'_{41} &= \Big[(1-b)y, \frac{B(y)}{(1-b)y}, \frac{aA(y)}{aby+1}, l_2(y)\Big]],\\
\partial_4^\infty Y'_{41} &= \Big[(1-b)x, \frac{ab-b+1}{b-1}, \frac{aA(x)}{abx+1}, \frac{ac-a}{ac-a+1}\Big],\\
\partial_5^0 Y'_{41} &= \Big[(1-b)x, \frac{B(y_2)}{(1-b)y_2}, \frac{aA(x)}{abx+1}, \frac{(ab-b+1)(y_2-x)}{(aby_2+1)B(x)}\Big].
\end{aligned}$$

All these cycles are clearly admissible. For the last one, we need (13) and (10).

$$\boxed{Y''_{41} = \Big[(1-b)x, \frac{B(y)}{(1-b)y}, b, \frac{(ab-b+1)(y-x)}{(aby+1)B(x)}, l_2(y)\Big]}$$

In the above proof for Y'_{41} we didn't use $t_3 = 1$ so the same proof is still valid.

$$\boxed{Y'_{42} = \Big[(1-b)x, \frac{B(y)}{(1-b)y}, \frac{abA(x)}{abx+1}, \frac{aby+1}{aA(y)}, l_2(y)\Big]}$$

This is a product of two admissible cycles. From the above we get

$$Y'_{41} = Y'_{41} + Y'_{42} = Y'_{41} + Y'_{42} + Y''_{41} = -Y'_4.$$

Claim 8.2 is proved.

Claim 8.3. Under non-degeneracy assumption

$$Y'_3 + Y'_4 = Y_3 + Y_4.$$

First it is not hard to show all the following cycles are admissible and negligible:

$$\begin{aligned}
Y_{31} =& \Big[-1, (b-1)y, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l_2(y)\Big],\\
Y_{41} =& \Big[(b-1)x, -1, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l_2(y)\Big],\\
Y_{32} =& \Big[\frac{B(x)}{(1-b)x}, -1, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l_2(y)\Big],\\
Y_{42} =& \Big[-1, \frac{B(y)}{(1-b)y}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l_2(y)\Big].
\end{aligned}$$

Then by Lemma 3.2(ii) we have

$$\begin{aligned}
Y_3'+Y_4' =&\Big[\frac{B(x)}{(1-b)x},(b-1)y,\frac{abx+1}{aA(x)},\frac{\mu(x-y)}{A(y)B(x)},l_2(y)\Big]+Y_{32}\\
&+\Big[(b-1)x,\frac{B(y)}{(1-b)y},\frac{abx+1}{aA(x)},\frac{\mu(x-y)}{A(y)B(x)},l_2(y)\Big]+Y_{42}\\
=&Y_3+Y_4+Y_{31}+Y_{41}\\
=&Y_3+Y_4.
\end{aligned}$$

Claim 8.3 is proved.

Step (9). Final decomposition of $\{k(c)\}$ into $T_i(F)$'s.

$$\boxed{T_1(A)=\Big[A(x),A(y),\frac{x-1}{x},\frac{y-x}{A(y)},\varepsilon_1(A)l_1(y)\Big],\quad \varepsilon_1(A)=\frac{ac}{ac-a+1}}$$

We have

$$\begin{aligned}
&\partial_1^0T_1(A)\subset\{t_4=1\},\quad \partial_1^\infty T_1(A)\subset\{t_3=1\},\quad \partial_2^0T_1(A)\subset\{t_5=1\},\\
&\partial_2^\infty T_1(A)\subset\{t_4=1\},\quad \partial_4^\infty T_1(A)\subset\{t_5=1\},\quad \partial_5^\infty T_1(A)\subset\{t_4=1\},
\end{aligned}$$

and

$$\begin{aligned}
\partial_3^0T_1(A)&=\Big[\frac{1}{a},A(y),\frac{y-1}{A(y)},\varepsilon_1(A)l_1(y)\Big],\\
\partial_3^\infty T_1(A)&=\Big[\frac{1-a}{a},A(y),\frac{y}{A(y)},\varepsilon_1(A)l_1(y)\Big],\\
\partial_4^0T_1(A)&=\Big[A(y),A(y),\frac{y-1}{y},\varepsilon_1(A)l_1(y)\Big],\\
\partial_5^0T_1(A)&=\Big[A(x),A(c),\frac{x-1}{x},\frac{c-x}{A(c)}\Big].
\end{aligned}$$

All these cycles are clearly admissible.

$$\boxed{T_2(A)=\Big[A(x),A(y),\frac{x-1}{x},\frac{y-x}{A(y)},\varepsilon_2(A)l_2(y)\Big],\quad \varepsilon_2(A)=\frac{ac-a+1}{ac}}$$

The above proof mostly is still valid because $\varepsilon_2(A)l_2((a-1)/a)=1$ except that

$$\begin{aligned}
\partial_5^0T_2(A)&=\Big[A(x),A(y_2),\frac{x-1}{x},\frac{y_2-x}{A(y_2)}\Big],\\
\partial_5^\infty T_2(A)&=\Big[A(x),\frac{\mu}{b-1},\frac{x-1}{x},\frac{B(x)}{-\mu}\Big]
\end{aligned}$$

which are both admissible.

$$\boxed{T_3(A)=\Big[A(x),A(y),\frac{abx+1}{abA(x)},\frac{y-x}{A(y)},\varepsilon_1(A)l_1(y)\Big],\quad \varepsilon_1(A)=\frac{ac}{ac-a+1}}$$

We have

$$\partial_1^0 T_3(A) \subset \{t_4 = 1\}, \quad \partial_1^\infty T_3(A) \subset \{t_3 = 1\}, \quad \partial_2^0 T_3(A) \subset \{t_5 = 1\}, \quad \partial_2^\infty T_3(A) \subset \{t_4 = 1\},$$
$$\partial_3^\infty T_3(A) \subset \{t_4 = 1\}, \quad \partial_4^\infty T_3(A) \subset \{t_5 = 1\}, \quad \partial_5^\infty T_3(A) \subset \{t_4 = 1\},$$

and

$$\begin{aligned}
\partial_3^0 T_3(A) &= \Big[\frac{\mu}{b}, A(y), \frac{aby+1}{abA(y)}, \varepsilon_1 l_1(y)\Big],\\
\partial_4^0 T_3(A) &= \Big[A(y), A(y), \frac{aby+1}{abA(y)}, \varepsilon_1 l_1(y)\Big],\\
\partial_5^0 T_3(A) &= \Big[A(x), A(c), \frac{abx+1}{abA(x)}, \frac{c-x}{A(c)}\Big].
\end{aligned}$$

All these cycles are clearly admissible.

$$\boxed{T_4(A) = \Big[A(x), A(y), \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, \varepsilon_2(A) l_1(y)\Big], \quad \varepsilon_2(A) = \frac{ac-a+1}{ac}}$$

The above proof mostly is still valid because $\varepsilon_2 l_2((a-1)/a) = 1$ except that

$$\begin{aligned}
\partial_5^0 T_4(A) &= \Big[A(x), A(y_2), \frac{abx+1}{abA(x)}, \frac{y_2-x}{A(y_2)}\Big],\\
\partial_5^\infty T_4(A) &= \Big[A(x), \frac{\mu}{b-1}, \frac{abx+1}{abA(x)}, \frac{B(x)}{-\mu}\Big]
\end{aligned}$$

which are both admissible.

Next we need to prove

$$\begin{aligned}
Z_3\Big(\frac{A}{f}, \frac{A}{f}\Big) &= \Big[\frac{A(x)}{x}, \frac{A(y)}{y}, \frac{x-1}{x}, \frac{y-x}{yB(x)}, l(y)\Big] = \Big[\frac{A(x)}{x}, \frac{A(y)}{y}, \frac{x-1}{x}, 1-\frac{x}{y}, l(y)\Big]\\
&= \Big[\frac{A(x)}{x}, \frac{A(y)}{y}, \frac{(1-a)(x-1)}{x}, \frac{y-x}{yB(x)}, l(y)\Big].
\end{aligned}$$

$$\boxed{Z_3'(A) = \Big[\frac{A(x)}{x}, \frac{A(y)}{y}, \frac{x-1}{x}, \frac{y-x}{y}, l(y)\Big]}$$

The only non-trivial boundaries are

$$\begin{aligned}
\partial_3^0 Z_3'(A) &= \Big[\frac{1-a}{a}, \frac{A(y)}{y}, \frac{y-1}{y}, l(y)\Big],\\
\partial_4^0 Z_3'(A) &= [\frac{A(y)}{y}, \frac{A(y)}{y}, \frac{y-1}{y}, l(y)\Big],\\
\partial_5^0 Z_3'(A) &= \Big[\frac{A(x)}{x}, \frac{A(c)}{c}, \frac{x-1}{x}, \frac{c-x}{c}\Big] + \Big[\frac{A(x)}{x}, \frac{A(y_2)}{y_2}, \frac{x-1}{x}, \frac{y_2-x}{y_2}\Big]
\end{aligned}$$

which are all admissible.

$$\boxed{Z_3''(A) = \Big[\frac{A(x)}{x}, \frac{A(y)}{y}, \frac{x-1}{x}, \frac{1}{B(x)}, l(y)\Big]}$$

It's admissible because $B(0) = 1$.

$$\boxed{Z_3'(A) = \Big[\frac{A(x)}{x}, \frac{A(y)}{y}, 1-a, \frac{y-x}{y}, l(y)\Big]}$$

It's admissible because $l(0) = 1$ and $l((a-1)/a) = 1$.

$$\boxed{T_1\Big(\frac{A}{f}\Big) = \Big[\frac{A(x)}{x}, \frac{A(y)}{y}, \frac{(1-a)(x-1)}{x}, 1-\frac{x}{y}, l_1(y)\Big]}$$

With the non-degeneracy assumption we have

$$\partial_1^0 T_1\Big(\frac{A}{f}\Big) \subset \{t_3 = 1\}, \quad \partial_1^\infty T_1\Big(\frac{A}{f}\Big) \subset \{t_4 = 1\}, \quad \partial_2^\infty T_1\Big(\frac{A}{f}\Big) \subset \{t_5 = 1\},$$
$$\partial_3^\infty T_1\Big(\frac{A}{f}\Big) \subset \{t_4 = 1\}, \quad \partial_4^\infty T_1\Big(\frac{A}{f}\Big) \subset \{t_5 = 1\}, \quad \partial_5^\infty T_1\Big(\frac{A}{f}\Big) \subset \{t_4 = 1\},$$

and

$$\begin{aligned}
\partial_2^0 T_1\Big(\frac{A}{f}\Big) &= \Big[\frac{A(x)}{x}, \frac{(1-a)(x-1)}{x}, \frac{aA(x)}{1-a}, \frac{ac-c+1}{ac}\Big],\\
\partial_3^0 T_1\Big(\frac{A}{f}\Big) &= \Big[\frac{1-a}{a}, \frac{A(y)}{y}, \frac{y-1}{y}, l_1(y)\Big],\\
\partial_4^0 T_1\Big(\frac{A}{f}\Big) &= \Big[\frac{A(y)}{y}, \frac{A(y)}{y}, \frac{(1-a)(y-1)}{y}, l_1(y)\Big],\\
\partial_5^0 T_1\Big(\frac{A}{f}\Big) &= \Big[\frac{A(x)}{x}, \frac{A(c)}{c}, \frac{(1-a)(x-1)}{x}, 1-\frac{x}{c}\Big].
\end{aligned}$$

All these cycles are clearly admissible.

$$\boxed{T_2\Big(\frac{A}{f}\Big) = \Big[\frac{A(x)}{x}, \frac{A(y)}{y}, \frac{(1-a)(x-1)}{x}, 1-\frac{x}{y}, l_2(y)\Big]}$$

The above proof mostly is still valid because $l_2(0) = 1$ except that

$$\begin{aligned}
\partial_5^0 T_2\Big(\frac{A}{f}\Big) &= \Big[\frac{A(x)}{x}, \frac{A(y_2)}{y_2}, \frac{(1-a)(x-1)}{x}, 1-\frac{x}{y_2}\Big],\\
\partial_5^\infty T_2\Big(\frac{A}{f}\Big) &= \Big[\frac{A(x)}{x}, -\mu, \frac{(1-a)(x-1)}{x}, B(x)\Big]
\end{aligned}$$

which are both admissible since $B(0) = 1$.

From (21) we have

$$[h, h, \delta v, q_4, l_1] = \Big[x, y, \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, l_1(y)\Big].$$

It is clear that
$$\left[x, y, \frac{abx+1}{abA(x)}, \frac{y}{A(y)}, l_1(y)\right]$$
is admissible by $l_1(0) = 1$. So we have
$$[h, h, \delta v, q_4, l_1] = \left[x, y, \frac{abx+1}{abA(x)}, \frac{y-x}{y}, l_1(y)\right]$$
which is also admissible.

Step (10). Final computation of $\{k(c)\}$.

We have shown in the above everything in this step is admissible. This completes the admissibility check of our main paper [Main].

References

[Main] A. B. Goncharov, *Goncharov's Relations in Bloch's higher Chow Group* $CH^3(F, 5)$, math.AG/0105084, to appear in J. Number Theory.